From Obstructed Ball Packing
To Symplectic Mapping Class Groups
& (hopefully) More...

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The Problem of ball Packing

It is a classical problem in symplectic geometry to determine whether it is possible to embed $\bigsqcup B(r_i) \hookrightarrow M$.

Why this is of particular interests?

Theorem (Gromov): $B(r) \hookrightarrow B(1) \times \mathbb{R}^2$ if and only if $r \leq 1$.

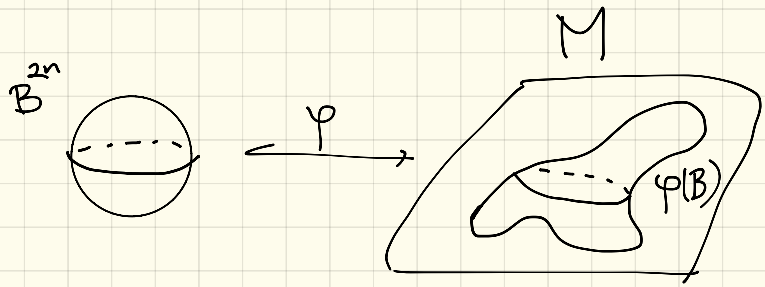
This shows symplectic geometry has its unique rigidity pheonomenon.

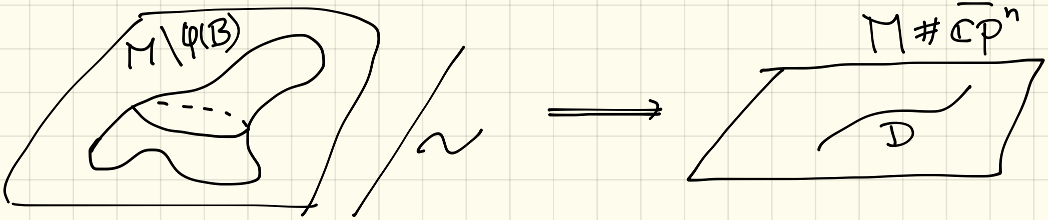
McDuff–Polterovich: ball-packing $\Longleftrightarrow$ symplectic form on blow-ups.
+ computed certain cases for packing $n \leq 8$ balls into a single ball

Biran: When $n \geq 9$, equal ball-packing has only volume obstruction.

Ball-packing vs. Symplectic blow-ups.

How to blow-up a symplectic mfd?





Relation $\sim$: S^1-orbits of Hopf fibration on $\partial\varphi(B^{2n})$

Remarks: 1) removing a point cannot endow $M \# \overline{\mathbb{C}P}^n$ a natural symp. form.

2) The above procedure is reversible.

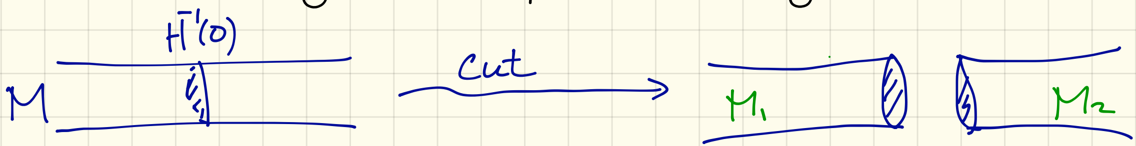
More general construction: Symplectic cuts.

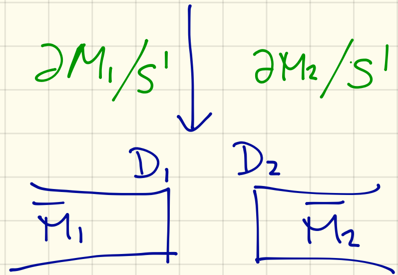
Given a Hamiltonian function on a neighborhood $\mathcal{U} \subset M$.

Suppose $H^{-1}(0) \subset \mathcal{U}$ is a closed manifold foliated by S^1- Ham. orbits, i.e. trajectories defined by Hamiltonian flow X_H def. as:

$$\omega(X_H, -) = dH$$

Then one may "cut open" M along $H^{-1}(0)$.





Ex: $M \xrightarrow{\text{cut}} M \# \overline{\mathbb{CP}^n}$ & $\mathbb{CP}^n$

Transforming a Packing Problem to symplectic forms:

Principle: Given a symplectic form on, for example, $\mathbb{CP}^2 \# n \overline{\mathbb{CP}}^2$, it uniquely (up to certain sense) correspond to a packing of n symplectic balls. The sizes are determined by ω-pairing with the exceptional divisors.

* The interests for understanding the blowing-up/down is also partly motivated by formulating a MMP in symplectic geometry. In dim = 4 this is understood after McDuff-Polterovich, Taubes-Li-Liu etc. but in higher dimensions many mysteries remain.

(one reason is we don't have bend-and-break!)

largely open

More Packing Objects : Non-equal sized balls , ellipsoids, Polydisks

Buse – Pinsonnault, etc.

McDuff

completely resolved

McDuff – Schlenk

+higher dim , ECH ···

A basic tool: Li – Liu wall-crossing formula and some explicit numerical reduction by Li – Li. Very special to dim = 4,

Taubes' SW $\Longleftrightarrow$ GW.

exclusive at the moment

* WILL MOSTLY FOCUS ON DIM = 4

IN THE REST OF THE TALK

Obstructed ball-packing v.s. Absolute ball-packing

How does the situation change if one wishes to add a submanifold obstruction to the packing problem?

NO EXTRA OBSTRUCTION ADDED FOR GW EFF. CLASSES.

Non-vanishing GW-invariant.

(Standard GW-techniques + positivity of intersection)

therefore, the interests focus on two kinds of obstructions:

- GW invariants = 0
- Lagrangian Submfds.

Why interesting:

Define $\mathcal{P}^L_{\bar{B}} = \{ \varphi : \bar{B} = \coprod_{\substack{i \in I \\ \text{finite}}} B(r_i) \hookrightarrow M \setminus L,\ L \text{ Lag} \}$

$\mathcal{L} = \{ \phi : L \hookrightarrow M^\# \text{ Lag. embedding},\ M^\# = \text{blow-up with size } r_i \}$

Expected relation between $\pi_i(\mathcal{P}^L_{\bar{B}})$ and $\pi_i(\mathcal{L})$.

Especially, the existence problem ($\pi_{-1}(?)$) and the Lagrangian uniqueness (π_0) are central topics in symplectic geometry.

$$\boxed{\text{WE WILL FOCUS ON THESE TWO PROBLEMS IN THE REST OF THE TALK}}$$

(1) Existence Problem:

Simplest case: $\mathbb{RP}^2 \subset \mathbb{CP}^2$, $\bar{\Delta} \subset S^2 \times S^2$

Biran: In $\mathbb{CP}^n$, packing obstructed by $\mathbb{RP}^n$ cannot be full.

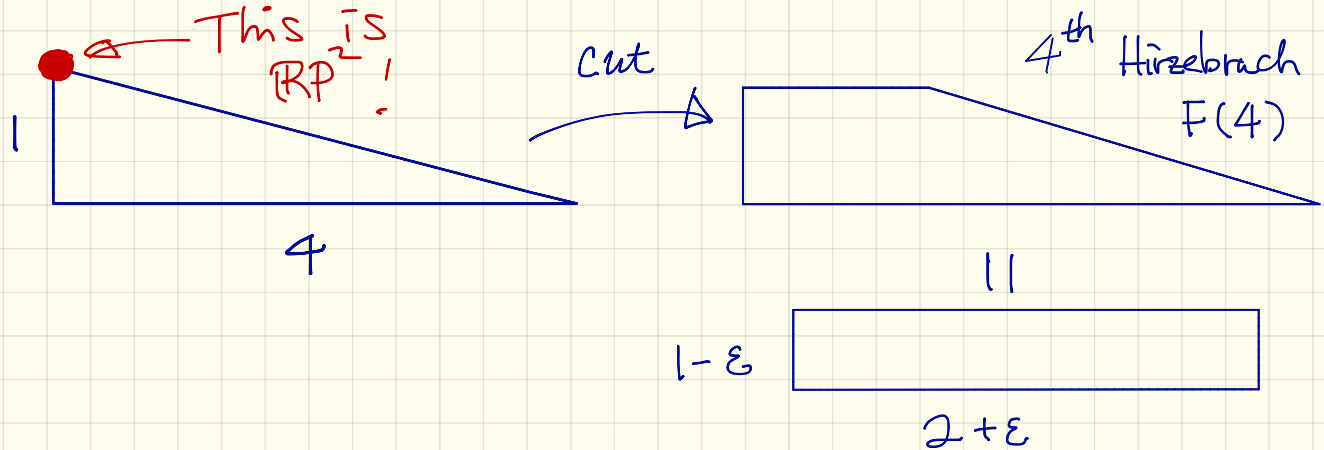
Full packing for $S^2 \times S^2$ obstructed by $\bar{\Delta}$.

Li-W: No extra obstruction for $\bar{\Delta}$, i.e. the obstructed packing problem is the same as the absolute one.

Same is true for blow-ups on $S^2 \times S^2$.

For $\mathbb{RP}^2$, we found the extra obstruction imposed by $\mathbb{RP}^2$ is "cosmetic".

A Toric Picture for $\mathbb{CP}^2 \setminus \mathbb{RP}^2$:



Theorem: (Borman-Li-W.) $\mathbb{RP}^2$-obstructed packing problem is equivalent to absolute packing in $S^2(1-\varepsilon) \times S^2(2+\varepsilon)$ when $\varepsilon \ll 1$.

Flipping the coin around:

Packing $\bigsqcup_n B(r_i) \hookrightarrow S^2 \times S^2 \setminus \bar{\Delta} \overset{\text{(trivial)}}{\Longrightarrow}$ can obtain $S^2 \times S^2 \# n\overline{\mathbb{CP}}^2$ with Lag. S^2 in $[\bar{\Delta}]$.

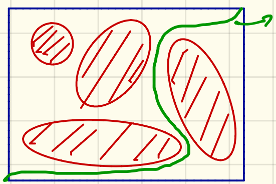
But converse also true!

This is again easy if all exceptional curves are arranged disjoint from L

class $= [\bar{\Delta}]$

$S^2 \times S^2 \# n\overline{\mathbb{CP}}^2$

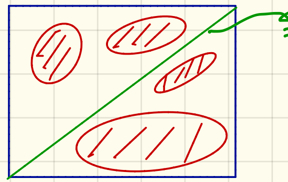
blow-down



class $= [\bar{\Delta}]$

$S^2 \times S^2$

$\exists$ Hamiltonian φ (Hind)



$= \bar{\Delta}$

$S^2 \times S^2$

Theorem (Li-W.) When $b^+(M^4)=1$, and GW-eff. class A has $\langle A, [L]\rangle = m \in \mathbb{Z}$, then there exists a symplectic representative Σ, s.t. $[\Sigma]=A$, $|\Sigma \cap L| = |m|$.

Especially, $\langle [E_i], [\bar{\Delta}]\rangle = 0$ always true $\Rightarrow$ $\exists$ disjoint representatives.

Original proof uses SFT, new proof McDuff-Opshtein's non-generic GW.

Slogan: Obstructed packing $\underset{\substack{\uparrow \\ \text{technical conditions}}}{\Longleftrightarrow}$ Existence of Lagrangian in blow-ups.

More obstructed problems:

Symplectic spheres: No extra $\Longleftrightarrow$ existence in blow-ups

Symp/Lag. config.: No extra, much more tricky.

[McDuff–Opshtein
Borman–Li–W.
Dorfmeister–Li–W.]

② Uniqueness Problem of Obstructed ball-Packing:

* In how many ways can you pack the balls? (Up to Ham isotopy)

McDuff: Only one when $b^+(M)=1$ in absolute packing.

Borman-Li-W.: Same is true for * Lag / Symp (-2) sphere obstruction.

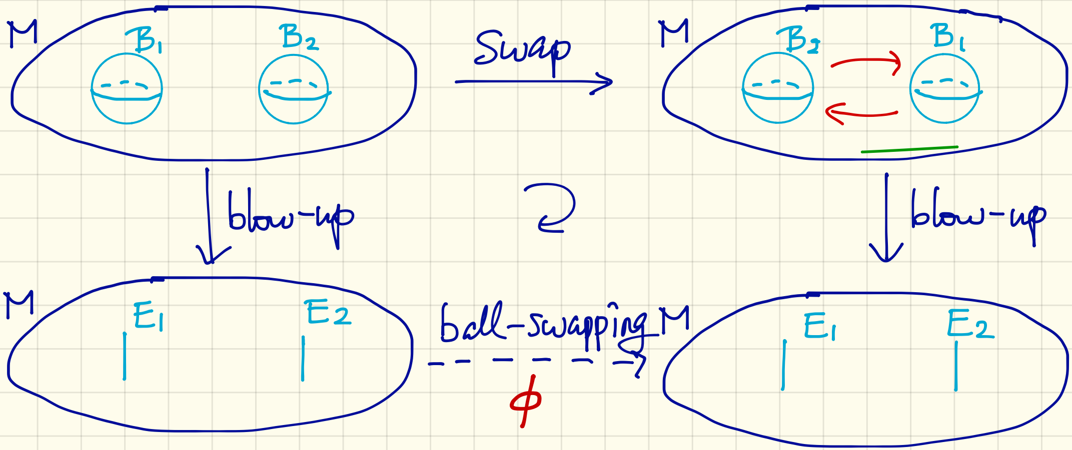
* Lag. $\mathbb{RP}^2$ / Symp. (-4) sphere

Relation to Lagrangian uniqueness, ball-swapping:

Theorem (BLW): Any homologous Lag. S^2 are related by symplectomorphisms in $(\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2, \omega)$, $\forall n$.

The case for Lag $\mathbb{RP}^2$ is similar in principle, but has tricky points that remains unsolved. So the parallel result is known only for $n \leq 8$.

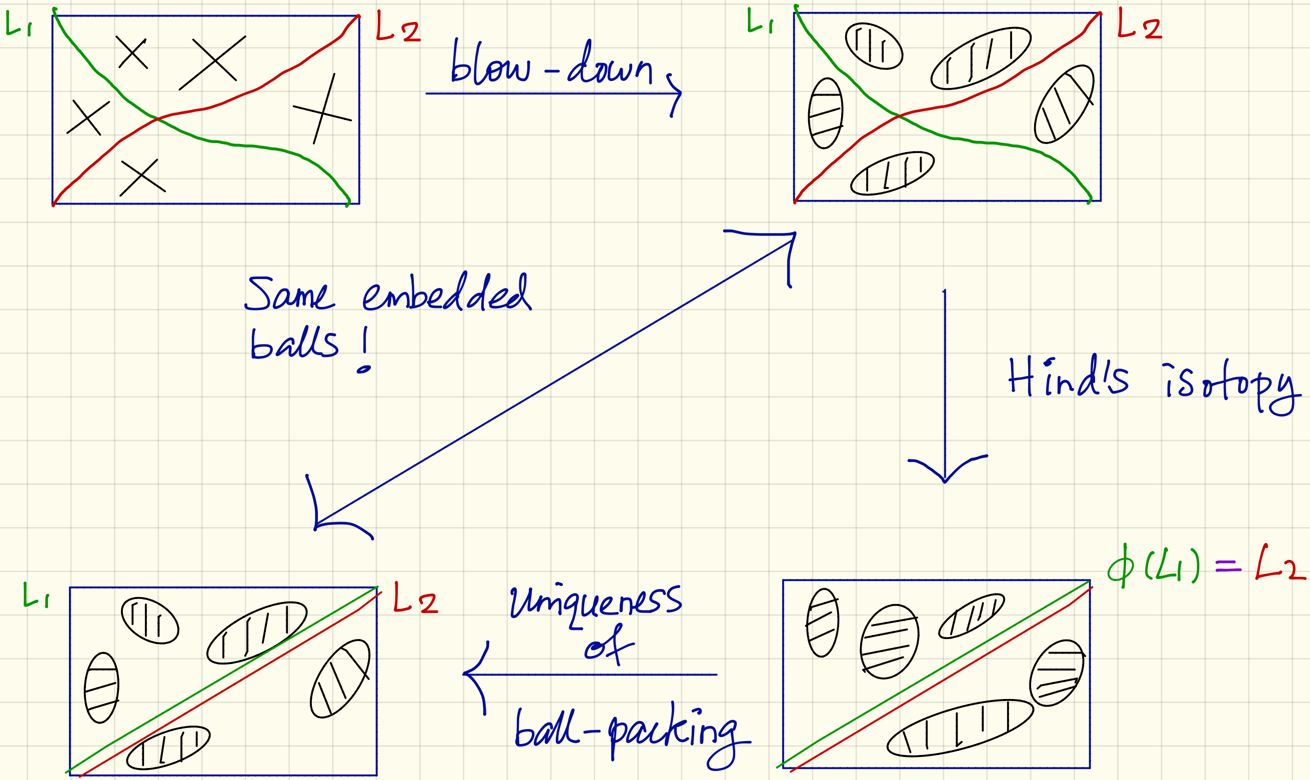
Ball-swapping: (local model)



$\phi^2 \in Symp_h(M \# 2\overline{\mathbb{CP}}^2)$

Clearly works as long as the two balls can be switched.

$\Leftarrow$ Uniqueness of ball packing. Also works with more balls.



L_1
L_2
blow-down
L_1
L_2
Same embedded balls!
Hind's isotopy
$\phi(L_1) = L_2$
uniqueness of ball-packing
L_1
L_2

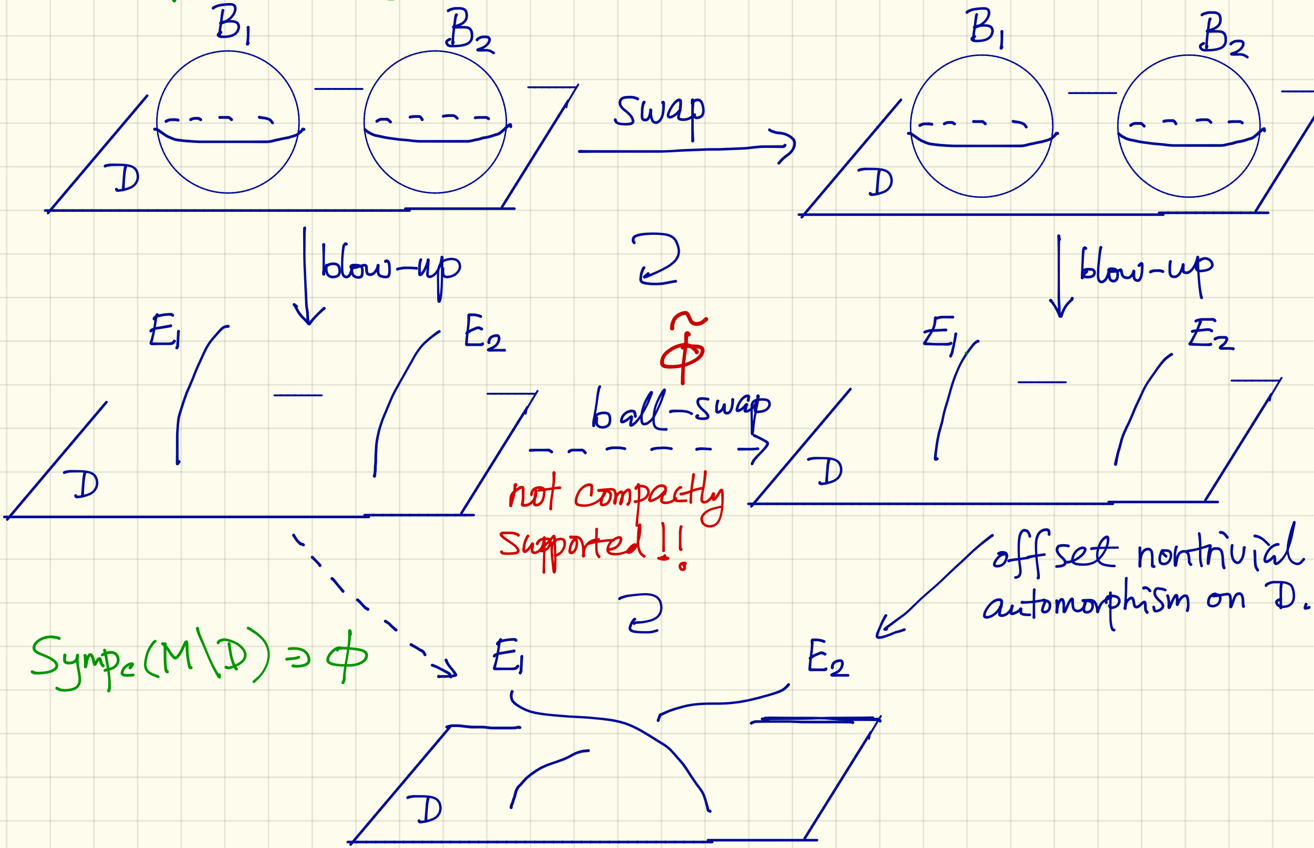
It is known in many cases ball-swapping = Dehn twists.

Proposition: $\pi_0(\mathrm{Symp}(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}))$ is generated by ball-swapping.

Q: Is this generated by Dehn twists?
($\Rightarrow$ for generic blow-up sizes $\pi_0(\mathrm{Symp}) = \{1\}$.)

* Can ball-swapping give info when no exceptional curves exist?

Non-compact analogue in $M \setminus D$:



$\mathrm{Symp}_c(M \setminus D) \ni \phi$

A_n-Milnor fiber: $\{(x,y,z)\in\mathbb{C}^3 \mid x^2+y^2+z^{n+1}=1\}$

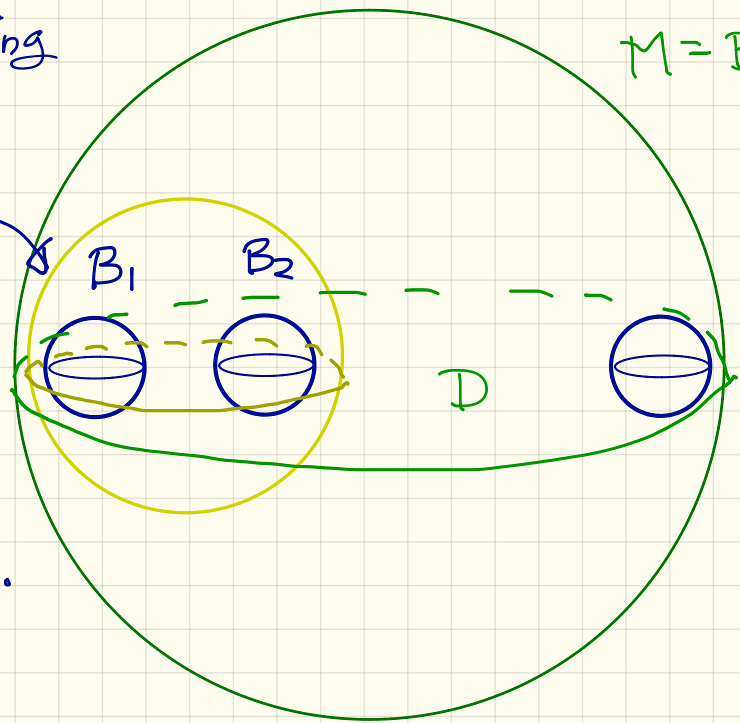
This is the symplectic plumbing of n-copies of T^*S^2.

Non-compact ball-swapping in local model

$\rightsquigarrow$ Generator of braid group.

$\Rightarrow \pi_0(\mathrm{Symp}_c(A_n)) = Br_{n+1}$

generated by Dehn twists.



Arnold's nearby Lagrangian Conjecture: If $L \hookrightarrow (T^*M, \omega_{std})$ as an exact Lagrangian, then L is Hamiltonian isotopic to zero section.

Beyond dimension 2, even to show L is homeomorphic to zero section is incredibly difficult.

Only proved for $M = S^2 / \mathbb{RP}^2$, NOT KNOWN FOR genus ≥ 1.

Fukaya-Seidel-Smith, Abouzaid $\Rightarrow$ homotopy equivalent.

Theorem (W.) Any two Lag. S^2 embedding in A_n-Milnor fiber are related by a composition of Dehn twists.

Some Further Questions:

1. $\pi_0(\mathrm{Symp}(\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}))$ gen. by Dehn twists?

2. Smooth and symplectic uniqueness of Lag. $\mathbb{RP}^2$.

3. Ball-swapping interpretation of Lag. $\mathbb{RP}^2$ Dehn twists?

4. Higher homotopy groups of obstructed ball-packing?

Closely related to topology of space of Lagrangians.

Known: ⊗ (Absolute) Lalonde-Pinsonnault.

⊗ space of Lag $S^2/\mathbb{RP}^2$ in $T^*S^2/T^*\mathbb{RP}^2/\mathbb{CP}^2/S^2\times S^2$.

(Hind, Hind-Pinsonnault-W.)