# Equivariant Split Generation & Mirror Symmetry of Some Nontrivial Surface Bundles.

## Rapid introduction to HMS:

| Symplectic Geom. (A-side) | Algebraic Geom. (B-side) |
| --- | --- |
| • Symplectic manifold M | • Alg. Objects { CY's, Alg. varieties, isolated sing. |
| • Fukaya Category | • Derived cat of { coherent sheaves, Matrix Factorizations |

## Kontsevich's HMS Conjecture:

$$D\,Fuk(X) \simeq D^b Coh(X)$$

Instances of proven cases:

quartic surfaces (Seidel), CY hypersurfaces (Sheridan)

closed symplectic surfaces (2 dim'l, Seidel - Efimov)

Punctured spheres (Abouzaid etc.)

Del Pezzo surfaces (Auroux etc.) ....

$(T^{2n}, \omega_{split})$ (Abouzaid - Smith) ← focus

Question: What about other symplectic forms on $T^{2n}$ ?

→ In general, the mirror symmetry effect on symplectic deformation could be very tricky. See [Auroux etc.]

# Linear symplectic Tori.

Given a symplectic form on $\mathbb{R}^{2n}$

$$\omega = \sum f_{ij}\, dx^i \wedge dx^j + \sum g_{ij}\, dx^i \wedge dy^j + \sum h_{ij}\, dy^i \wedge dy^j.$$

$$\omega^n > 0, \quad d\omega = 0.$$

Linear symplectic forms: $f_{ij}$, $g_{ij}$, $h_{ij}$ are constants.

Full lattices on $\mathbb{R}^{2n}$: $e_1, \cdots, e_{2n}$ are lin. independent over $\mathbb{R}$, then $\Gamma = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_{2n}$ is a full lattice. And $\mathbb{R}^{2n}/\Gamma$ is a $\mathbb{T}^{2n}$ smoothly.

Note: Shifting preserves lin. symplectic form

$\Rightarrow (\mathbb{R}^{2n}, \omega^{lin})/\Gamma$ is a symplectic $\mathbb{T}^{2n}$.

$\Rightarrow$ linear symplectic form on $\mathbb{T}^{2n}$.

(Folklore?) Conjecture:

Any symplectic forms on $\mathbb{T}^{2n}$ are symplectomorphic to a linear one.

Although currently, such a conjecture still remains clueless it gives us a guiding principle for restricting our attention to linear forms.

Example: $\mathbb{T}(\alpha) \times \mathbb{T}(\beta) \simeq (\mathbb{R}^4, \omega_{std}) / \langle (\alpha, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, \beta), (0, 0, 0, 1) \rangle$

Consider $G_{\alpha,\beta} = \omega(H_2(\mathbb{T}^{2n})) < \mathbb{R}$. If $\alpha, \beta, \alpha', \beta'$ are not rationally dependent, then

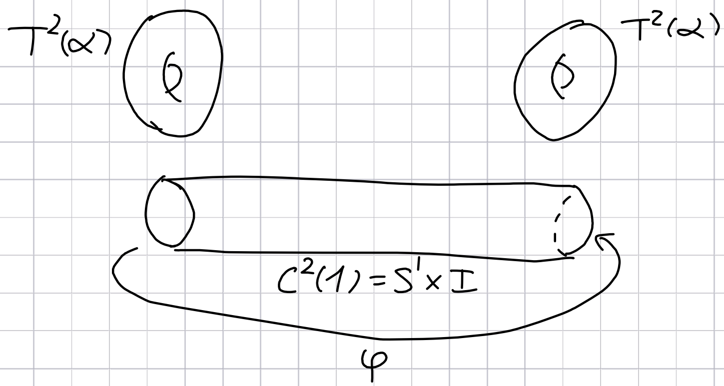
$G_{\alpha,\beta} \neq G_{\alpha',\beta'} \Rightarrow \mathbb{T}(\alpha) \times \mathbb{T}(\beta) \neq \mathbb{T}(\alpha) \times \mathbb{T}(\beta)$.

$G_{\alpha,\beta}$ can be considered as an invariant of symp. tori.

**Example:** $\mathbb{T}^2(\alpha) \times C^2(1) / \sim_\varphi \; := \mathbb{T}(\alpha, 1)_n$

$$\varphi : S^1(\alpha) \times S^1(1) \longrightarrow S^1(\alpha) \times S^1(1)$$

$$(a \;,\; b) \longmapsto (a + \tfrac{\alpha}{n} \;,\; b)$$




$$\mathbb{T}(\alpha,1)_n \simeq (\mathbb{R}^{2n}, \omega) / T^{s+d}, \qquad \omega = \left(\begin{array}{cc|cc} 0 & 0 & \alpha & -\frac{1}{n} \\ 0 & 0 & 0 & 1 \\ \hline -\alpha & 0 & 0 & 0 \\ \frac{1}{n} & -1 & 0 & 0 \end{array}\right)$$

$$\simeq \mathbb{T}^2(\alpha) \times \mathbb{T}^2(1) / (\mathbb{Z}/n)$$

This can be generalized:

Definition: (Symplectic Special isogenous tori)

$$\mathbb{T}^2(\alpha_1)\times\cdots\times\mathbb{T}^2(\alpha_n)/(\mathbb{Z}/\ell),\ \ell = \mathrm{lcm}(\ell_1,\ell_2,\cdots,\ell_n)$$

$$\mathbb{Z}/\ell_i = \langle g_i\rangle,\quad g_i : S^1(\alpha_i)\times S^1(1) \to S^1(\alpha_i)\times S^1(1)$$

$$(a,\quad) \mapsto (a+\alpha_i/\ell_i,\ b)$$

Lemma: (Latschev-McDuff-Schlenk)

If $\omega(H_2(\mathbb{T}^{2n}))<\mathbb{Q}$, then $(\mathbb{T}^{2n},\omega)\simeq\mathbb{T}(\alpha_1)\times\cdots\times\mathbb{T}(\alpha_n)$ (split)

Note: $\mathbb{T}(1,\alpha)n$ is not split when $\alpha$ is irrational.

**Theorem:** $D^{\pi}\mathrm{Fuk}(\mathbb{T}(\alpha_1, \alpha_2, \cdots, \alpha_n)_e)$

$$\simeq D^b(A(\alpha_1, \alpha_2, \cdots, \alpha_n)_e)$$

Some abelian variety

**Another direction of interests:**

* Any linear symplectic tori are symplectomorphic iff they are symplectomorphic by linear transformations.
  (take the cohomological map and use the fact that linear forms are unique in their classes)
* No symplectic invariants can distinguish them (absence of GW, Gromov width etc.)

**Theorem:** (Orlov) For $A_1, A_2$ abelian varieties over a field $\mathbb{k}$, $\mathrm{char}(\mathbb{k}) = 0$. $D^b(A_1) \simeq D^b(A_2) \Leftrightarrow A_1 \times \hat{A}_1 \underset{\text{isometric}}{\simeq} A_2 \times \hat{A}_2$

**Corollary:** Derived Fukaya Category is a complete invariant in the class of S.I. tori.

**Question:** Can mirror symmetry distinguish symplectic structures of related example?

First example is $T^*S^1 \times \Pi(\alpha_1, \cdots, \alpha_n)_e$.

**Conjecture:** $T^*S^1 \times \Pi(\alpha_1, \cdots, \alpha_n)_e \underset{\text{Symp}}{\cong} T^*S^1 \times \Pi(\alpha_1', \cdots, \alpha_n')_{e'}$

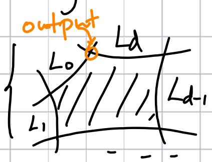
iff $\Pi(\alpha_1, \cdots, \alpha_n)_e \cong \Pi(\alpha_1', \cdots, \alpha_n')_{e'}$.

# Equivariant Fukaya Categories for finite free actions.

Setting: $M$ closed symplectic, $c_1(M,\omega)=0$, $\pi_2(M)=0$.
$L \subset M$ Lagrangians ($\omega|_L=0$), $\mu(L)=0$, spin.
$G \circlearrowright (M,\omega)$ symplectic, fix-point free.

| $Fuk(M)$ | $Fuk(M)^G$ |
|---|---|
| Objects: $L \subset M$ Lag. with spin structure and chosen grading. $(spin_L, gr^L)$ | $G\cdot L = \bigcup_{g\in G} g\cdot L$, $(L, spin_L, gr^L) \subset M$. $\forall\, g\cdot L$ equipped with push-forward auxilliary data $(g\cdot spin_L,\ g\cdot gr^L)$ |
| Mor: $L_1 \pitchfork L_2$, "good" choices of perturbation | $\left(\bigcup_{g_i, g_j \in G} g_i L_1 \pitchfork g_j L_2\right)^G$, equivariant "good" choices. |

| $Fuk(M)$ | $Fuk(M)^G$ |
| --- | --- |
| $A_\infty$-compositions: | |
| $\mu^d$: Rigid counts of $\mathcal{M}(y; x^d, \cdots, x^1)$ | $G$-orbits of $\mathcal{M}(y; x^d, \cdots x^1)$ |
| = | Here the choice of Floer data is a bit more delicate but manageable. |




$\Rightarrow$ $A_\infty$-categories $\left( \sum_{d_1+d_2=d} \pm \mu^{d_1}(\cdots, \mu^{d_2}(\cdots), \cdots) = 0 \right)$

$\xrightarrow[\text{manipulation}]{\text{alg.}}$ $D^b Fuk(M)^G$, a triangulated cat.

* A way to look at $Fuk(M)^G$ is that this is a subcat (non-full) of $D^b Fuk(M)$. Hence $D^b Fuk(M)^G$ is the fixed subcat. of the $G$-action.

**Generation**: $\{X_\alpha\}_\alpha \subset \mathcal{A}$ be a subset of objects. $\{X_\alpha\}$ split generates $K \in \mathcal{A}$ iff one gets $K$ after taking cones/shifts/direct summand within $\{X_\alpha\}$.

**Theorem** (Abouzaid-Smith) In $\mathbb{T}^2(\alpha_1) \times \cdots \times \mathbb{T}^2(\alpha_n)$, Lagrangians of the form $L_1^{m(\ell)} \times L_2^{m(\ell)} \cdots \times L_n^{m(\ell)}$ split generates $Fuk(\mathbb{T}^{2n}(\alpha_1, \cdots, \alpha_n))$.

**Theorem**: (W.-) If $G \curvearrowright (M, \omega)$ free and finite, then $\exists\ \mathcal{J}: Fuk(X/G) \longrightarrow Fuk(X)^G$ which is fully faithful. Moreover, if $\{L_\alpha\}$ split generates $Fuk(M)$, s.t. $L_\alpha/G \subset M/G$ are embedded, then $\mathcal{J}$ is an equivalence.

* One checks that this is the case for $M = \mathbb{T}^{2n}(\alpha_1, \cdots, \alpha_n)$, and $M/G = \mathbb{T}^{2n}(\alpha_1, \cdots, \alpha_n)_\ell$, $\{L_\alpha\}$ be prod. Lag. as above.

## Homological Mirror Symmetry.

$A(\alpha_1,\cdots,\alpha_n)$
$\parallel$
$\vdots$

**Theorem** (Abouzaid–Smith)

$$D^b\mathrm{Fuk}(\mathbb{T}^2(\alpha_1)\times\cdots\times\mathbb{T}^2(\alpha_n)) \simeq D^b\mathrm{Coh}(\Lambda^*/\langle q^{\alpha_1}\rangle\times\cdots\times\Lambda^*/\langle q^{\alpha_n}\rangle)$$

**Note**: The $\mathbb{Z}/\ell$-action on $\mathbb{T}(\alpha_1,\cdots,\alpha_n) \Rightarrow \mathbb{Z}/\ell$-action on $D^\pi\mathrm{Fuk}(\mathbb{T}(\alpha_1\cdots\alpha_n))$

$\Rightarrow$ — // — $D^b(\Lambda^*/\langle q^{\alpha_1}\rangle\times\cdots\times\Lambda^*/\langle q^{\alpha_n}\rangle)$

**Theorem**: (W.–)

already proved

$$D^b(\mathrm{Fuk}(\mathbb{T}(\alpha_1\cdots\alpha_n)))^{\mathbb{Z}/\ell} \simeq D^b(\mathrm{Fuk}(\mathbb{T}(\alpha_1\cdots\alpha_n)/(\mathbb{Z}/\ell)))$$

Some alg. reduction $\rightarrow$ $\wr\|$ $\qquad\qquad$ $\|\wr$

$$D^b(\Lambda^*/\langle q^{\alpha_1}\rangle\times\cdots\times\Lambda^*/\langle q^{\alpha_n}\rangle)^{\mathbb{Z}/\ell} \simeq D^b(A(\alpha_1,\cdots,\alpha_n)_\ell)$$

classical

**Theorem** (Comparison of Fuk on B-side)

$$D^bA(\alpha_1,\cdots,\alpha_n)_\ell \simeq D^bA(\alpha_1'\cdots\alpha_n')_{\ell'}$$

$\Leftrightarrow \mathbb{T}(\alpha_1,\cdots,\alpha_n)_\ell$ symplectomorphic to $\mathbb{T}(\alpha_1'\cdots\alpha_n')_{\ell'}$

The last comparison uses:

1) $A(\alpha_1 \cdots \alpha_n)$ is an abelian variety (Riemann condition)

2) Orlov's criterion

3) If $A = (\Lambda^*)^n / T$, then $\hat{A} = \mathrm{Hom}(T, \Lambda^*) / \mathrm{Hom}((\Lambda^*)^n, \Lambda^*)$

THANKS FOR YOUR ATTENTION!