Global Surfaces of Section

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Outline

The 3-body problem

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The equations (written in [Sun13]) that govern the 3 Body problem are

$$\begin{cases} \dfrac{dx_0}{dt} = x'_0, & \dfrac{dx'_0}{dt} = m_1 \dfrac{x_1 - x_0}{r_2^3} + m_2 \dfrac{x_2 - x_0}{r_1^3}, \\ \dfrac{dy_0}{dt} = y'_0, & \dfrac{dy'_0}{dt} = m_1 \dfrac{y_1 - y_0}{r_2^3} + m_2 \dfrac{y_2 - y_0}{r_1^3}, \\ \dfrac{dz_0}{dt} = z'_0, & \dfrac{dz'_0}{dt} = m_1 \dfrac{z_1 - z_0}{r_2^3} + m_2 \dfrac{z_2 - z_0}{r_1^3}, \end{cases}$$

The 3-body problem

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\begin{aligned}
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\frac{dx_2}{dt} &= x'_2, & \frac{dx'_2}{dt} &= m_0 \frac{x_0 - x_2}{r_1^3} + m_1 \frac{x_1 - x_2}{r_0^3}, \\
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  corresponding to zero angular momentum, a solution can locally be written in terms of a power series in a new time coordinate τ.
- The new time coordinate and radius of convergence can be explicitly given in terms of

$$m_i, x_i(0), y_i(0), z_i(0), x_i'(0), y_i'(0), z_i'(0), i \in \{1, 2, 3\}$$

  The convergence is slow beyond practical.

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- The way to **simplify** the problem is to add adjectives: Restricted, Planar, Circular, Regularized.
- Poincaré was on a quest of periodic orbits for the RP3BP and had a genius idea. Transfer 3D to 2D!

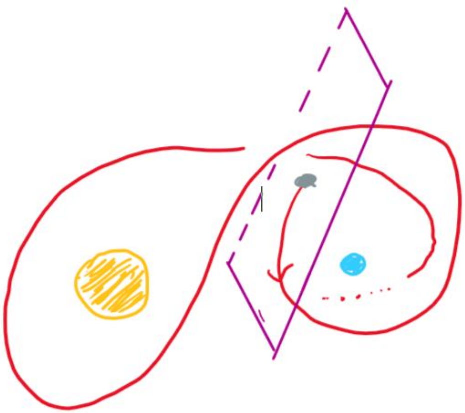

Figure: RP3BP.

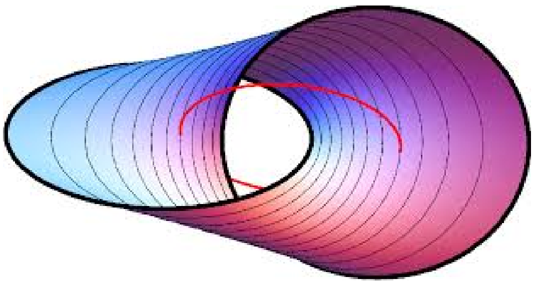

Figure: The purple surface turns out to be an annulus. Found in [Moe17]

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- If someone needs to find periodic orbits, they need to search for **sections** Σ, i.e. surfaces with boundary. The flow has to be transverse to their interior and their boundary should consist of periodic orbits.
- **Global** sections Σ, i.e. **for all points** on the manifold, either their trajectory following the flow intersects the interior transversely both in the future and the past or the trajectory of the point is on the boundary of the surface. (Section has enough info.)
- Poincaré then explained that a proof to the following theorem would provide periodic orbit, i.e. solutions to the RP3BP.

Poincaré's/Birkhoff's work.

Theorem (Poincaré-Birkhoff)

A homeomorphism of the closed annulus

- *Area-preserving*
- *Orientation-preserving*
- *Twisting its boundary components in opposite directions*

has at least two fixed points.

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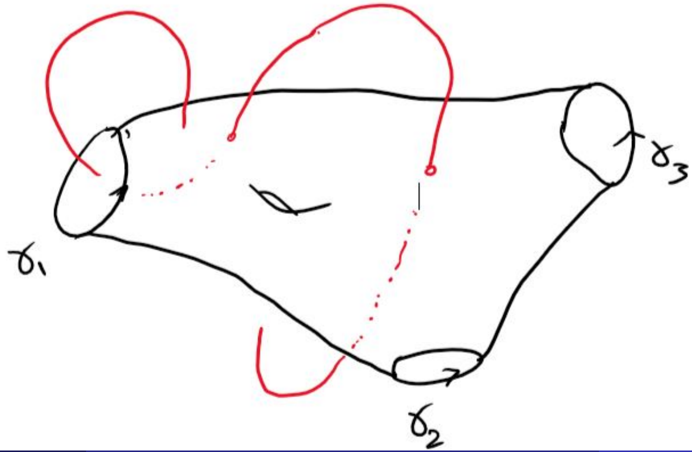
- Poincaré published the proof in special cases for consideration by other mathematicians.
- A year later, Birkhoff in [Bir13] provided a full proof.

Definitions

Definition

Let ϕ^t be a smooth flow on a smooth closed 3-dimensional manifold M. An embedded surface $\Sigma \hookrightarrow M$ is a global surface of section for the flow ϕ^t if:

- Each component of $\partial\Sigma$ is a periodic orbit of ϕ^t.
- ϕ^t is transverse to $\Sigma \backslash \partial\Sigma$
- Globality: For every $p \in M \backslash \partial\Sigma$ there exist times $t_+ > 0$ and $t_- < 0$ such that $\phi^{t_+}(p), \phi^{t_-}(p) \in \Sigma \backslash \partial\Sigma$



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For every $p \in \Sigma \backslash \partial\Sigma$, we call $\tau(p) := \inf\{t > 0 \mid \phi^t(p) \in \Sigma\}$ the first return time.

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- We can think of M as a regular energy level of the Hamiltonian system.
- Later we will focus on a special case of 3-dimensional flows, namely Reeb flows in 3-dimensional contact manifolds.
- Upshot: Dynamics are encoded in the first return map. Thus, GSS discretize the flow and reduce dimensions.

Example 0: Geodesic flow on S^2

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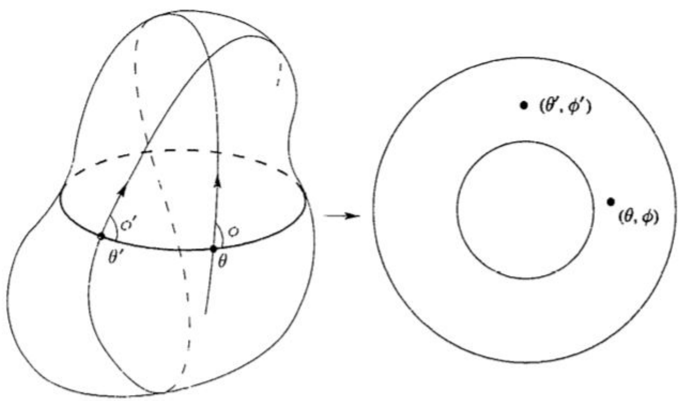
- Also, consider the annulus $[0, \pi] \times [0, \pi]$. The first angle refers to a specific point on the equator and the second to a vector at the point of the equator pointing towards the upper hemisphere or the equator in either direction.
- This Annulus, called the **Birkhoff Annulus** records the initial conditions of a geodesic starting at a point on the equator. The following figure is from [Cip93].



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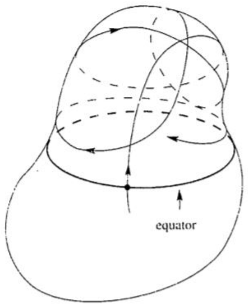
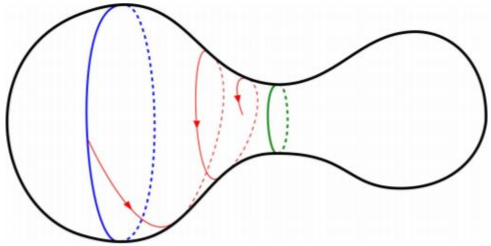
- <u>Birkhoff:</u> Curvature $K > 0 \Rightarrow$ the return map is well defined.
- This can fail in other cases as the following pictures illustrate. The first is from [Cip93] and the second from [Oan14].





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Case 1:

- Birkhoff's map is area preserving.
- Franks: For any area preserving homeomorphism of the annulus there are either 0 or ∞-many periodic points.
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Case 2:

- Bangert handled the case when the return map to the Birkhoff annulus fails to exist with differential geometry.

Contact Geometry

Definition

Let M^3 be a closed manifold.

- $(M^3, \xi = \ker \alpha)$ where $\alpha \wedge d\alpha$ is a volume form is called a contact manifold.
- The 1-form α is called a contact form.

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Definition

The Reeb vector field associated to the contact 1-form α is the vector field R_α determined by

- $\iota_{R_\alpha} d\alpha = 0$
- $\alpha(R_\alpha) = 1$

Its flow is called the Reeb flow on M.

Example 1: Standard S^3

- Consider $S^3 := \{(z_1, z_2) \in \mathbb{C}^2 || z_1|^2 + |z_2|^2 = 1\}$ inside $\mathbb{C}^2$ with its standard primitive for the symplectic form

$$\begin{aligned} \lambda_0 &= \frac{i}{2}(z_1 d\overline{z}_1 - \overline{z}_1 dz_1 + z_2 d\overline{z}_2 - \overline{z}_2 dz_2) \\ &= \frac{1}{2}\sum_{j=1}^{2} x_j dy_j - y_j dx_j \end{aligned}$$

where we set $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

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- Set $\alpha_0 := \lambda_0|_{S^3}$
- It is easy to check that $\alpha_0 \wedge d\alpha_0$ is a volume form on S^3.
- This makes (S^3, α_0) a contact manifold.

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- $R_{\alpha_0} = 2\sum_{j=1}^{2} x_j\partial_{y_j} - y_j\partial_{x_j}$ and its flow is found by the ODE system

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- We get $R_{\alpha_0} = 2(\partial_{\theta_1} + \partial_{\theta_2})$ and the system becomes

$$\dot{\theta}_1 = 2$$
$$\dot{\theta}_2 = 2$$

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- Thus, $\phi^t(z_1, z_2) = (e^{2it}z_1, e^{2it}z_2)$, i.e the flow rotates each factor of $\mathbb{C}^2$ separately with the same angular speed for each factor.

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- For its complement, we have the diffeomorphism
  $g : \text{Int}(\mathbb{D}) \times \mathbb{R}/\pi\mathbb{Z} \to S^3$, with $g(r, \theta, s) = (re^{i(\theta+2s)}, \sqrt{1-r^2}e^{2is})$

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- We see $R_{\alpha_0} = \partial_s$, so the flow is everywhere transverse to the meridian disks.

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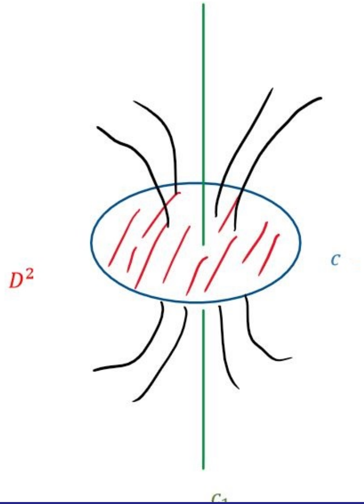
- The next two points are particularly useful for Jun's talk.
- Any disk bounded by C in this foliation is a global surface of section.
- The return map of such disk is the identity which is obviously symplectic and the return time is π.
- Thus, there are plenty of global surfaces of sections in S^3.



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- Conclusion: Naturally one expects a general theorem for such space.
- This is the main theorem from [HWZ98] for global surface of sections on convex hypersurfaces in $\mathbb{R}^4$ containing the origin. It is discussed in a few slides.
- Before this though, we present an example which demonstrates that even in simple examples, where the flow is completely understood explicitly, it is not easy to decide whether a global surface of section exists.

Example 2: Standard torus

- Consider $T^3 := (\mathbb{R}/2\pi\mathbb{Z})_\theta \times (\mathbb{R}^2/\mathbb{Z}^2)_{x,y}$

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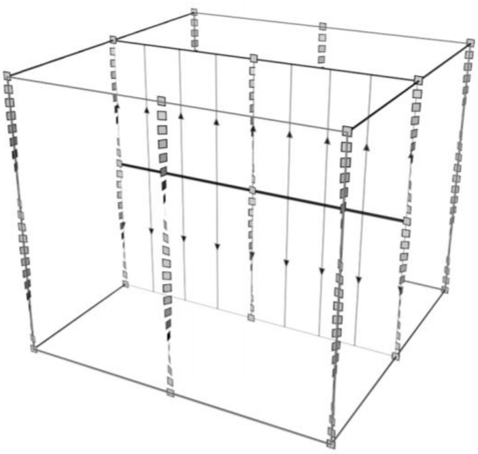
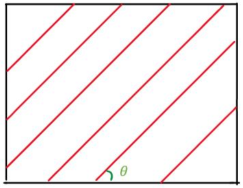


Figure: Contact structure on T^3. Found in Patrick Massot's website.

- All orbits are horizontal so a global surface of sections should not be horizontal.



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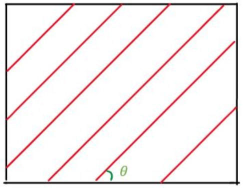

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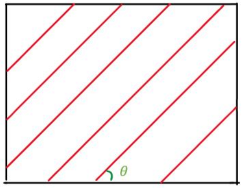

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- A good candidate for a global surface of section would be some 2-dimensional torus corresponding to fixed y_0, $T^2 = \{(\theta, x, y_0)\}$.
- There is a $(0,1)$-worth of Reeb orbits missing it, precisely when $\theta = 0$ and 1 orbit which is not transverse.

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- Maybe if we “rotate”, “bend” or “add genus” to this vertical plane we get a corresponding closed surface and a global surface of section exists.
- It seems hard to find a global surface of section even in this simple explicit example.
- Yet topological considerations in this case do not seem to provide an obstruction. (e.g. T^3 fibers over S^1).
- Hence, finding a global surface of section is generally a hard problem.

Main Existence Theorem

The following is a special case of theorem 1.3 in [HWZ98].

Theorem (Hofer-Wysocki-Zehnder)

Let M be a convex hypersurface in $\mathbb{R}^4$ enclosing the origin, equipped with $\alpha = \lambda_0|_M$. Let R_α be the associated Reeb vector field. Then $\exists$ periodic orbit P_0 of R_α s.t.

- *P_0 is the boundary of a disk-like global surface of section $\mathcal{D}$.*
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- This theorem, as well as the next one, holds in the more general case of dynamically convex contact forms. (See Jean's talk.)

Existence Theorems

The following theorem is again a special case of a theorem found in [Hry14].

Theorem (Hryniewicz)

Consider a convex hypersurface M in $\mathbb{R}^4$ enclosing the origin, equipped with the contact form $\alpha = \lambda_0|_M$.

- *A periodic orbit γ bounds a disk-like global surface of section $\Leftrightarrow$ It is unknotted and has self-linking number -1.*
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- This unknotted orbit in our main example is C.
- The equivalence in the theorem yields room for obstructions.

Some Obstructions

Frauenfelder and Van Koert discuss the following obstructions in their book [FvK18]:

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- **Obstruction 2:** If a periodic orbit is the binding orbit of a global surface of section, it is linked to every other periodic orbit.
- **Obstruction 3:** If a periodic Reeb orbit γ is the binding orbit of a disk-like global surface of section, then its self-linking number satisfies $sl(\gamma) = -1$.

Non-Existence theorem of disk-like Surface of Section

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Non-Existence theorem of disk-like Surface of Section

- The question of existence of a Reeb flow on the tight S^3 without any global surface of sections is still open.
- In [vK20], Van Koert using a certain Murasugi sum construction (connected sum of abstract open books), shows that obstruction 2 holds providing the following partial result.

Theorem (Van Koert)

- *There is a Reeb flow on (S^3, ξ_0) that does not admit a global surface of section with only one boundary component.*
- *In particular, this Reeb flow does not admit a disk-like global surface of section.*

Non-Existence theorem of disk-like Surface of Section

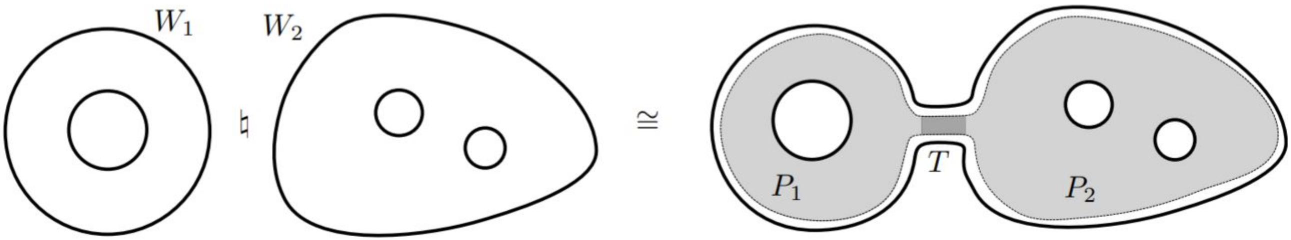

Figure: Murasugi Sum. Found in [vK20]

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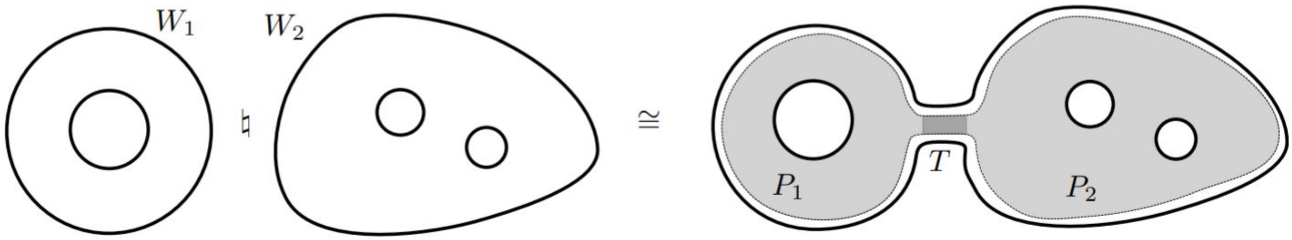

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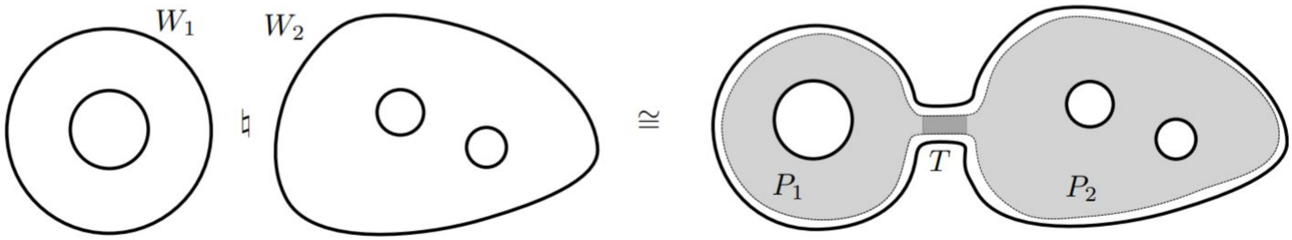

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- Van Koert chooses 3 Open Book Decompositions for the standard tight S^3 and performs the Murasugi sum of them.
- He equips the sum with a specific Reeb flow such that the colored regions below are invariant sets.

Non-Existence theorem of disk-like Surface of Section

- The specific sum is described in the following picture.

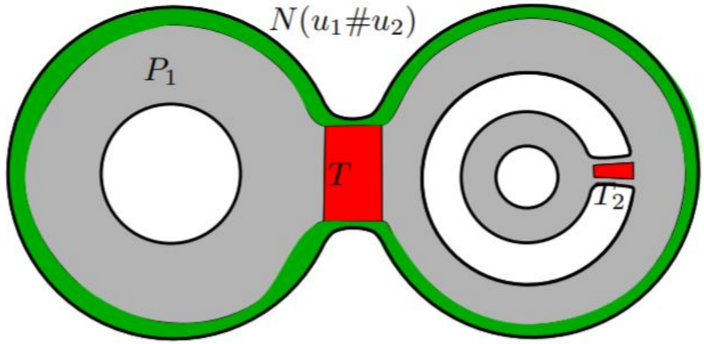

Figure: The Murasugi Sum in the proof.

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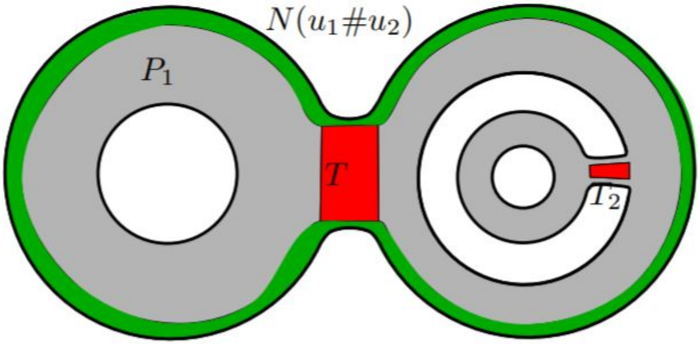

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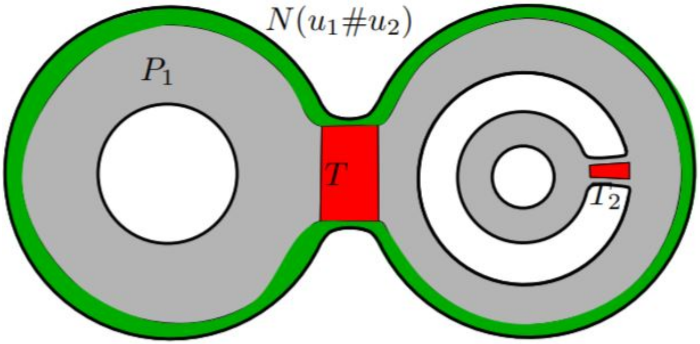

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- The invariant regions contain Reeb orbits which do not link to the orbits of any other region.
- By obstruction 2, we need more boundary components in order to be linked with the Reeb orbits of all invariant regions.

Thank you!

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