Algebraic K-theory in symplectic geometry

Jun Zhang

CRM - Université de Montréal

July 21, 2020

Motivation

- The main objects that will be discussed in this talks are

$$K_0 \text{ group of a category } \; K_0(\cdot)$$

and

$$\text{Hochschild homology of a category } \; \mathrm{HH}_*(\cdot)$$

Motivation

- The main objects that will be discussed in this talks are

$$K_0 \text{ group of a category } \; K_0(\cdot)$$

and

$$\text{Hochschild homology of a category } \; \mathrm{HH}_*(\cdot)$$

- The motivation of this talk is to understand the map f_2 in the following picture,

$$K_0(D\mathcal{F}uk^*(M)) \xrightarrow{f_1} K_0(\mathcal{Y}(\mathcal{F}uk^*(M))^\wedge)$$
$$\xrightarrow{f_2} HH_*(\mathcal{Y}(\mathcal{F}uk^*(M))^\wedge) \xrightarrow{f_3} HH_*(\mathcal{F}uk^*(M)) \xrightarrow{f_4} QH(M).$$

from Page 2695 in Biran-Cornea's *Cone-decompositions of Lagrangian cobordisms in Lefschetz fibrations* (Selecta 2017).

K_0 group of an abelian category

• Algebraic K-theory has a complicated history. Giants were involved: A. Grothendieck, H. Bass, J. Milnor, D. Quillen, $\cdots$.

K_0 group of an abelian category

• Algebraic K-theory has a complicated history. Giants were involved: A. Grothendieck, H. Bass, J. Milnor, D. Quillen, $\cdots$.

• Let $\mathscr{A}$ be an abelian category, so one can discuss isomorphism classes and short exact sequences.

Definition

$K_0(\mathscr{A})$ is the abelian group freely generated by the isomorphism classes of objects of $\mathscr{A}$, denoted by $[A]$ for $A \in \mathrm{Obj}(\mathscr{A})$, modulo the relation that $[B] = [A] + [C]$ if and only if there exists the following short exact sequence in $\mathscr{A}$

$$0 \to A \to B \to C \to 0.$$

K_0 group of an abelian category

• Algebraic K-theory has a complicated history. Giants were involved: A. Grothendieck, H. Bass, J. Milnor, D. Quillen, $\cdots$.

• Let $\mathscr{A}$ be an abelian category, so one can discuss isomorphism classes and short exact sequences.

Definition

$K_0(\mathscr{A})$ is the abelian group freely generated by the isomorphism classes of objects of $\mathscr{A}$, denoted by $[A]$ for $A \in \mathrm{Obj}(\mathscr{A})$, modulo the relation that $[B] = [A] + [C]$ if and only if there exists the following short exact sequence in $\mathscr{A}$

$$0 \to A \to B \to C \to 0.$$

Example

Let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$*, the category of* $\mathbf{k}$*-modules. Then* $K_0(\mathscr{A}) \simeq \mathbb{Z}$*. Indeed, for any* $x = m_1[A_1] + ... + m_n[A_n] = [V] - [W]$ *in* $K_0(\mathscr{A})$*, consider the isomorphism* $\dim_{\mathbf{k}} : K_0(\mathscr{A}) \to \mathbb{Z}$ *defined by*

$$\dim_{\mathbf{k}}(x) = \dim_{\mathbf{k}}(V) - \dim_{\mathbf{k}}(W).$$

Triangulated category

An abelian category $\mathscr{C}$ is called **triangulated** if there exists an automorphism $T : \mathscr{C} \to \mathscr{C}$ and *triangles* as follows,

$$\Delta : \ A \to B \to C \to T(A);$$

where there is a set of *distinguished triangles* $\mathrm{dist}_{\Delta}(\mathscr{C})$ satisfying

Triangulated category

An abelian category $\mathscr{C}$ is called **triangulated** if there exists an automorphism $T : \mathscr{C} \to \mathscr{C}$ and *triangles* as follows,

$$\Delta : \; A \to B \to C \to T(A);$$

where there is a set of *distinguished triangles* $\mathrm{dist}_\Delta(\mathscr{C})$ satisfying

(TR0) If $\Delta' \simeq \Delta$ for $\Delta \in \mathrm{dist}_\Delta(\mathscr{C})$, then $\Delta' \in \mathrm{dist}_\Delta(\mathscr{C})$. For any $A \in \mathrm{Obj}(\mathscr{C})$, $A \xrightarrow{e_A} A \to 0 \to A[1] \in \mathrm{dist}_\Delta(\mathscr{C})$.

(TR1) Any morphism $f : A \to B$ can embed into a $\Delta \in \mathrm{dist}_\Delta(\mathscr{C})$.

(TR2) If $\Delta \in \mathrm{dist}_\Delta(\mathscr{C})$, then $T(\Delta), T^{-1}(\Delta) \in \mathrm{dist}_\Delta(\mathscr{C})$.

(TR4) The octahedral axiom.

$$\begin{array}{ccccc}
E & \longrightarrow & 0 & \longrightarrow & T(E) \\
\downarrow & & \downarrow & & \vdots \\
F & \overset{g\circ f}{\dashrightarrow} & A & \dashrightarrow & C \\
{\scriptstyle f}\downarrow & & \downarrow & & \vdots \\
X & \overset{g}{\longrightarrow} & A & \longrightarrow & B
\end{array}$$

Example I

Example

Let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$. *Denote by* $\mathrm{Hot}^b(\mathscr{A})$ *the homotopy category of bounded complexes of* **k**-*modules.*

$$\mathrm{Obj}(\mathrm{Hot}^b(\mathscr{A})) = \{A_\bullet \,|\, A_\bullet \textit{ is a bounded chain complex}\}$$

and $\mathrm{Hom}_{\mathrm{Hot}^b(\mathscr{A})} = \{\textit{chain maps}\}/\sim$, *where* $f \sim g$ *means* f *and* g *are homotopic.*

Example I

Example

Let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$. *Denote by* $\mathrm{Hot}^b(\mathscr{A})$ *the homotopy category of bounded complexes of* $\mathbf{k}$*-modules.*

$$\mathrm{Obj}(\mathrm{Hot}^b(\mathscr{A})) = \{A_\bullet \,|\, A_\bullet \textit{ is a bounded chain complex}\}$$

and $\mathrm{Hom}_{\mathrm{Hot}^b(\mathscr{A})} = \{\textit{chain maps}\}/\sim$, *where* $f \sim g$ *means* f *and* g *are homotopic. Then* $\mathrm{Hot}^b(\mathscr{A})$ *is a triangulated category. In particular* $T(A_\bullet) = A_\bullet[1] = A_{\bullet-1}$.

Example

Let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$*. Denote by* $\mathrm{Hot}^b(\mathscr{A})$ *the homotopy category of bounded complexes of* $\mathbf{k}$*-modules.*

$$\mathrm{Obj}(\mathrm{Hot}^b(\mathscr{A})) = \{A_\bullet \,|\, A_\bullet \textit{ is a bounded chain complex}\}$$

and $\mathrm{Hom}_{\mathrm{Hot}^b(\mathscr{A})} = \{\textit{chain maps}\}/\sim$*, where* $f \sim g$ *means* f *and* g *are homotopic. Then* $\mathrm{Hot}^b(\mathscr{A})$ *is a triangulated category. In particular* $T(A_\bullet) = A_\bullet[1] = A_{\bullet-1}$.

Usually there are two variants based on the example above.

Example (Variant I)

Let $\mathscr{D}(\mathrm{Mod}(\mathbf{k}))$ *be a derived category of* $\mathrm{Mod}(\mathbf{k})$*. By definition,*

$$\mathscr{D}^b(\mathrm{Mod}(\mathbf{k})) = \mathrm{Hot}^b(\mathscr{A})[\{\mathrm{quasimorphisms}\}^{-1}]$$

i.e., quasi-isomorphisms are invertible. A famous example is $\mathscr{D}^b(\mathrm{Coh}X)$*, derived category of coherent sheaves on a smooth projective variety* X.

Example II

Example (Variant II)

A differential graded category (for brevity, called **dg-category***) $\mathscr{C}$ is a category consisting of a set of objects $\mathrm{Obj}(\mathscr{C})$ and for any $A, B \in \mathrm{Obj}(\mathscr{C})$,*

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \text{ is a chain complex of } \mathbf{k}\text{-modules}$$

such that composition $\circ$ is a chain map (Leibniz rule).

Example II

Example (Variant II)

A differential graded category (for brevity, called **dg-category***) $\mathscr{C}$ is a category consisting of a set of objects* $\mathrm{Obj}(\mathscr{C})$ *and for any* $A, B \in \mathrm{Obj}(\mathscr{C})$,

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \text{ is a chain complex of } \mathbf{k}\text{-modules}$$

such that composition $\circ$ *is a chain map (Leibniz rule).*

- *The degree-0 closed element in* $\mathrm{Hom}_{\mathscr{C}}(A, B)$ *is a chain map. The image of* ∂_1 *are null homotopy elements.*

Example II

Example (Variant II)

A differential graded category (for brevity, called **dg-category***)* $\mathscr{C}$ *is a category consisting of a set of objects* $\mathrm{Obj}(\mathscr{C})$ *and for any* $A, B \in \mathrm{Obj}(\mathscr{C})$,

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \text{ is a chain complex of } \mathbf{k}\text{-modules}$$

such that composition $\circ$ *is a chain map (Leibniz rule).*

- *The degree-0 closed element in* $\mathrm{Hom}_{\mathscr{C}}(A, B)$ *is a chain map. The image of* ∂_1 *are null homotopy elements.*
- *Replace* $\mathrm{Hom}_{\mathscr{C}}(A, B)$ *by its homology group, we obtain the* **homology category** *of* $\mathscr{C}$ *denoted by* $\mathrm{Ho}_\bullet(\mathscr{C})$*. In particular,*

$$\mathrm{Hom}_{\mathrm{Ho}_0(\mathscr{C})}(A, B) \in \mathrm{Hot}^b(\mathrm{Mod}(\mathbf{k})),$$

  where $\mathrm{Ho}_0(\mathscr{C})$ *is called the* **homotopy category** *of* $\mathscr{C}$*.*

Example II

Example (Variant II)

A differential graded category (for brevity, called **dg-category***) $\mathscr{C}$ is a category consisting of a set of objects* $\mathrm{Obj}(\mathscr{C})$ *and for any* $A, B \in \mathrm{Obj}(\mathscr{C})$,

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \text{ is a chain complex of } \mathbf{k}\text{-modules}$$

such that composition $\circ$ *is a chain map (Leibniz rule).*

- *The degree-0 closed element in* $\mathrm{Hom}_{\mathscr{C}}(A,B)$ *is a chain map. The image of* ∂_1 *are null homotopy elements.*
- *Replace* $\mathrm{Hom}_{\mathscr{C}}(A,B)$ *by its homology group, we obtain the* **homology category** *of* $\mathscr{C}$ *denoted by* $\mathrm{Ho}_\bullet(\mathscr{C})$. *In particular,*

$$\mathrm{Hom}_{\mathrm{Ho}_0(\mathscr{C})}(A,B) \in \mathrm{Hot}^b(\mathrm{Mod}(\mathbf{k})),$$

where $\mathrm{Ho}_0(\mathscr{C})$ *is called the* **homotopy category** *of* $\mathscr{C}$.

Remark

A dg-category is a special case of an A_∞-category (with operators $\{\mu_i\}_{i\geq 1}$) where $\mu_1 =$ differential, $\mu_2 =$ composition, and $\mu_{\geq 3} = 0$.

About (TR1)

In $\mathrm{Hot}^b(\mathscr{A})$ above, to complete $f_\bullet : A_\bullet \to B_\bullet$ into a distinguished triangle, we use **mapping cone** $\mathrm{Cone}_\bullet(f)$ defined as $B_\bullet \oplus A_\bullet[1]$ with its k-th differential defined by

$$d_k^{\mathrm{co}} := \begin{pmatrix} d_k^B & -f_{k-1} \\ 0 & d_{k-1}^A \end{pmatrix} : B_k \oplus A_k[1] \to B_{k-1} \oplus A_{k-1}[1].$$

About (TR1)

In $\mathrm{Hot}^b(\mathscr{A})$ above, to complete $f_\bullet : A_\bullet \to B_\bullet$ into a distinguished triangle, we use **mapping cone** $\mathrm{Cone}_\bullet(f)$ defined as $B_\bullet \oplus A_\bullet[1]$ with its k-th differential defined by

$$d_k^{\mathrm{co}} := \begin{pmatrix} d_k^B & -f_{k-1} \\ 0 & d_{k-1}^A \end{pmatrix} : B_k \oplus A_k[1] \to B_{k-1} \oplus A_{k-1}[1].$$

Therefore, we get a distinguished triangle,

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{i_B} \mathrm{Cone}_\bullet(f) \xrightarrow{\pi_A} A_\bullet[1].$$

Remark

The requirement of (TR0) implies the necessity to work over homotopy category (instead of the category of chain complexes).

About (TR1)

In $\mathrm{Hot}^b(\mathscr{A})$ above, to complete $f_\bullet : A_\bullet \to B_\bullet$ into a distinguished triangle, we use **mapping cone** $\mathrm{Cone}_\bullet(f)$ defined as $B_\bullet \oplus A_\bullet[1]$ with its k-th differential defined by

$$d_k^{\mathrm{co}} := \begin{pmatrix} d_k^B & -f_{k-1} \\ 0 & d_{k-1}^A \end{pmatrix} : B_k \oplus A_k[1] \to B_{k-1} \oplus A_{k-1}[1].$$

Therefore, we get a distinguished triangle,

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{i_B} \mathrm{Cone}_\bullet(f) \xrightarrow{\pi_A} A_\bullet[1].$$

Remark

The requirement of (TR0) implies the necessity to work over homotopy category (instead of the category of chain complexes).

Exercise

The morphism $f_\bullet : A_\bullet \to B_\bullet$ is an isomorphism in $\mathrm{Hot}^b(\mathscr{A})$, then $\mathrm{Cone}_\bullet(f) \simeq 0$ in $\mathrm{Hot}^b(\mathscr{A})$.

K_0 group of a triangulated category

Definition

Let $\mathscr{C}$ be a triangulated category. $K_0(\mathscr{C})$ is an abelian group freely generated by the isomorphism classes of objects of $\mathscr{C}$, denotes by $[A]$ for $A \in \mathrm{Obj}(\mathscr{C})$, modulo the relation $[B] = [A] + [C]$ if and only if $A \to B \to C \to T(A) \in \mathrm{dist}_{\Delta}(\mathscr{C})$.

K_0 group of a triangulated category

Definition

Let $\mathscr{C}$ be a triangulated category. $K_0(\mathscr{C})$ is an abelian group freely generated by the isomorphism classes of objects of $\mathscr{C}$, denotes by $[A]$ for $A \in \mathrm{Obj}(\mathscr{C})$, modulo the relation $[B] = [A] + [C]$ if and only if $A \to B \to C \to T(A) \in \mathrm{dist}_\Delta(\mathscr{C})$.

Remark

The first conclusion of (TR0) implies the definition is well-defined.

K_0 group of a triangulated category

Definition

Let $\mathscr{C}$ be a triangulated category. $K_0(\mathscr{C})$ is an abelian group freely generated by the isomorphism classes of objects of $\mathscr{C}$, denotes by $[A]$ for $A \in \mathrm{Obj}(\mathscr{C})$, modulo the relation $[B] = [A] + [C]$ if and only if $A \to B \to C \to T(A) \in \mathrm{dist}_\Delta(\mathscr{C})$.

Remark

The first conclusion of (TR0) implies the definition is well-defined.

Example

For any $A \in \mathrm{Obj}(\mathscr{C})$, the second conclusion of (TR0) and (TR2) implies that $A \to 0 \to T(A) \to T(A) \in \mathrm{dist}_\Delta(\mathscr{C})$. Therefore,

$$0 := [0] = [A] + [T(A)] \ \text{ in } K_0(\mathscr{C}),$$

which says $[T(A)] = -[A]$. This is how "inverse" appears in $K_0(\mathscr{C})$.

K_0 group of a triangulated category

Definition

Let $\mathscr{C}$ be a triangulated category. $K_0(\mathscr{C})$ is an abelian group freely generated by the isomorphism classes of objects of $\mathscr{C}$, denotes by $[A]$ for $A \in \mathrm{Obj}(\mathscr{C})$, modulo the relation $[B] = [A] + [C]$ if and only if $A \to B \to C \to T(A) \in \mathrm{dist}_\Delta(\mathscr{C})$.

Remark

The first conclusion of (TR0) implies the definition is well-defined.

Example

For any $A \in \mathrm{Obj}(\mathscr{C})$, the second conclusion of (TR0) and (TR2) implies that $A \to 0 \to T(A) \to T(A) \in \mathrm{dist}_\Delta(\mathscr{C})$. Therefore,

$$0 := [0] = [A] + [T(A)] \text{ in } K_0(\mathscr{C}),$$

which says $[T(A)] = -[A]$. This is how "inverse" appears in $K_0(\mathscr{C})$.

In $K_0(\mathrm{Hot}^b(\mathscr{A}))$, $[\mathrm{Cone}_\bullet(f)] = [B_\bullet] - [A_\bullet]$.

K_0 group computation

Naive but basic questions:

K_0 group computation

Naive but basic questions:

- Given a triangulated category $\mathscr{C}$, what is $K_0(\mathscr{C})$?
- Given an additive category, what is $K_0(\mathrm{Hot}^b(\mathscr{A}))$?
- In K_0 group, we have seen inverse $-[A]$ and subtraction $[B]-[A]$. How about the following alternating sum

$$[A]-[B]+[C]-[D]+\cdots \text{ ?}$$

- In K_0 group, how about the following linear combination

$$[X_0]+[X_1]+\cdots+[X_n] \text{ ?}$$

- How about subgroups of K_0 group, for instance,

$$K_0(\mathscr{C}) \simeq K_0(\mathscr{C}_1) \oplus K_0(\mathscr{C}_2) \oplus \cdots \oplus K_0(\mathscr{C}_n).$$

- $\cdots$

Reduction

Theorem

Let $\mathscr{A}$ *be an abelian category. Then* $K_0(\mathscr{D}^b(\mathscr{A})) \simeq K_0(\mathscr{A})$.

Reduction

Theorem

Let $\mathscr{A}$ be an abelian category. Then $K_0(\mathscr{D}^b(\mathscr{A})) \simeq K_0(\mathscr{A})$.

I am going to demonstrate a low-tech version: let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$, we have $K_0(\mathrm{Hot}^b(\mathscr{A})) \simeq K_0(\mathscr{A}) = \mathbb{Z}$.

Reduction

Theorem

Let $\mathscr{A}$ be an abelian category. Then $K_0(\mathscr{D}^b(\mathscr{A})) \simeq K_0(\mathscr{A})$.

I am going to demonstrate a low-tech version: let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$, we have $K_0(\mathrm{Hot}^b(\mathscr{A})) \simeq K_0(\mathscr{A}) = \mathbb{Z}$.

In order to prove this, consider the following "magic" map $\chi : K_0(\mathrm{Hot}^b(\mathscr{A})) \to K_0(\mathscr{A})$ defined by (Euler characteristic)

$$\chi([A_\bullet]) := \sum_{k=-\infty}^{\infty} (-1)^k [A_k].$$

Reduction

Theorem

Let $\mathscr{A}$ be an abelian category. Then $K_0(\mathscr{D}^b(\mathscr{A})) \simeq K_0(\mathscr{A})$.

I am going to demonstrate a low-tech version: let $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$, we have $K_0(\mathrm{Hot}^b(\mathscr{A})) \simeq K_0(\mathscr{A}) = \mathbb{Z}$.

In order to prove this, consider the following "magic" map $\chi : K_0(\mathrm{Hot}^b(\mathscr{A})) \to K_0(\mathscr{A})$ defined by (Euler characteristic)

$$\chi([A_\bullet]) := \sum_{k=-\infty}^{\infty} (-1)^k [A_k].$$

This map χ is well-defined (partially) due to the following exercise.

Exercise

If $A_\bullet \simeq B_\bullet$ in $\mathrm{Hot}^b(\mathscr{A})$, then $\sum(-1)^k[A_k] = \sum(-1)^k[B_k]$ in $K_0(\mathscr{A})$.

Sketch of the proof

Recall tha map

$$\chi([A_\bullet]) = \sum_{k=-\infty}^{\infty} (-1)^k [A_k] : \ K_0(\mathrm{Hot}^b(\mathscr{A})) \to K_0(\mathscr{A}).$$

- The map is a homomorphism, and obviously it is surjective.
- This map is also injective, that is, given $[A_\bullet]$ such that $\chi([A_\bullet]) = 0$, we claim that $[A_\bullet] = 0$ in $K_0(\mathrm{Hot}^b(\mathscr{A}))$. For instance, consider

$$A_\bullet = \cdots \to 0 \to A_1 \xrightarrow{d_1^A} A_0 \to 0 \to \cdots.$$

The assumption $\chi([A_\bullet]) = 0$ implies that $[A_1] = [A_0]$ in $K_0(\mathscr{A})$. Consider degree-0 centered complexes

$$X_\bullet = \cdots \to 0 \to A_1 \to 0 \to \cdots \ \text{ and } \ Y_\bullet = \cdots \to 0 \to A_0 \to 0 \to \cdots.$$

Then we have a distinguished triangle, $X_\bullet \xrightarrow{-d_1^A} Y_\bullet \to A_\bullet \to X_\bullet[1]$ since $A_\bullet = \mathrm{Cone}(-d_1^A)$. Therefore,

$$[A_\bullet] = [\mathrm{Cone}(-d_1^A)] = [Y_\bullet] - [X_\bullet] = 0 \ \text{ in } K_0(\mathrm{Hot}^b(\mathscr{A})).$$

Iterated cone decomposition

Definition

Let $\mathscr{C}$ be a triangulated category, and $X \in \text{Obj}(\mathscr{C})$. An **iterated cone decomposition** D **of** X **with linearization** $(X_0, X_1, ..., X_n)$ where $X_i \in \text{Obj}(\mathscr{C})$ consists of a family of distinguished triangles

$$\begin{cases} \Delta_1 : & T^{-1}(X_1) \to X_0 \to Y_1 \to X_1 \\ \Delta_2 : & T^{-1}(X_2) \to Y_1 \to Y_2 \to X_2 \\ & \vdots \\ \Delta_{n-1} : & T^{-1}(X_{n-1}) \to Y_{n-2} \to Y_{n-1} \to X_n \\ \Delta_n : & T^{-1}(X_n) \to Y_{n-1} \to X \to X_n \end{cases}$$

For brevity, a linearization in a cone decomposition D is a denoted by $\ell(D) = (X_0, X_1, ..., X_n)$.

Iterated cone decomposition

Definition

Let $\mathscr{C}$ be a triangulated category, and $X \in \mathrm{Obj}(\mathscr{C})$. An **iterated cone decomposition** D **of** X **with linearization** $(X_0, X_1, ..., X_n)$ where $X_i \in \mathrm{Obj}(\mathscr{C})$ consists of a family of distinguished triangles

$$\begin{cases} \Delta_1 : & T^{-1}(X_1) \to X_0 \to Y_1 \to X_1 \\ \Delta_2 : & T^{-1}(X_2) \to Y_1 \to Y_2 \to X_2 \\ & \vdots \\ \Delta_{n-1} : & T^{-1}(X_{n-1}) \to Y_{n-2} \to Y_{n-1} \to X_n \\ \Delta_n : & T^{-1}(X_n) \to Y_{n-1} \to X \to X_n \end{cases}$$

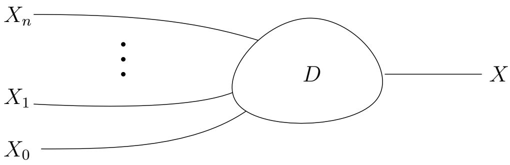
For brevity, a linearization in a cone decomposition D is a denoted by $\ell(D) = (X_0, X_1, ..., X_n)$.

Passing to $K_0(\mathscr{C})$,

$$[X] = [Y_{n-1}] + [X_n] = ([Y_{n-2}] + [X_{n-1}]) + [X_n] = ... = \sum_{i=0}^{n} [X_n].$$

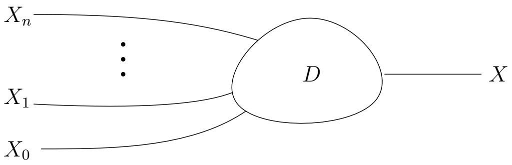
Geometric picture

Here is a pictorial way to represent this relation in $K_0(\mathscr{C})$.



Geometric picture

Here is a pictorial way to represent this relation in $K_0(\mathscr{C})$.



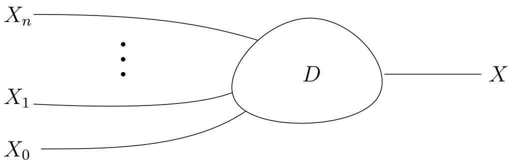
Geometric meaning of the picture above - Lagrangian cobordism!

Theorem (Biran-Cornea)

Suppose $(V; L_0 \cup \ldots \cup L_n, L)$ *is a Lagrangian cobordism from* L *to* $(L_0, \ldots, L_n)$ *in* M*, then there exist* $X_0, \ldots, X_n \in \mathrm{Obj}(\mathscr{D}\mathrm{Fuk}(M))$ *with* $X_0 = L_0$ *and* $X \simeq L$ *such that* V *provides a cone decomposition* D *of* X *with linearization* $(X_0, \ldots, X_n)$.

Geometric picture

Here is a pictorial way to represent this relation in $K_0(\mathscr{C})$.



Geometric meaning of the picture above - Lagrangian cobordism!

Theorem (Biran-Cornea)

Suppose $(V; L_0 \cup \ldots \cup L_n, L)$ is a Lagrangian cobordism from L to $(L_0, \ldots, L_n)$ in M, then there exist $X_0, \ldots, X_n \in \mathrm{Obj}(\mathscr{D}\mathrm{Fuk}(M))$ with $X_0 = L_0$ and $X \simeq L$ such that V provides a cone decomposition D of X with linearization $(X_0, \ldots, X_n)$.

Remark (Biran-Cornea-Shelukhin)

One can study the size of D (**shadow**), which provides a quantitative study of Lagrangian cobordisms.

Algebraic fragmentation pseudo-metric

Assume we can define $w(D) \in \mathbb{R}_{\geq 0}$, **weight of a cone decomposition**. One can give the following definition.

Algebraic fragmentation pseudo-metric

Assume we can define $w(D) \in \mathbb{R}_{\geq 0}$, **weight of a cone decomposition**. One can give the following definition.

Definition

Let $\mathscr{F} \subset \mathrm{Obj}(\mathscr{C})$ be a family of objects in $\mathscr{C}$. For two objects $X, X' \in \mathrm{Obj}(\mathscr{C})$, define

$$\delta^{\mathscr{F}}(X, X') := \inf \left\{ w(D) \;\middle|\; \begin{array}{l} D \text{ is an iterated cone decomposition} \\ \text{of } X \text{ (in } \mathscr{C}\text{) with a linearization as} \\ \text{with } \ell(D) = (F_0, F_1, ..., X', ..., F_k) \\ \text{with the objects } F_i \in \mathscr{F},\ k \in \mathbb{N}. \end{array} \right\}.$$

Algebraic fragmentation pseudo-metric

Assume we can define $w(D) \in \mathbb{R}_{\geq 0}$, **weight of a cone decomposition**. One can give the following definition.

Definition

Let $\mathscr{F} \subset \mathrm{Obj}(\mathscr{C})$ be a family of objects in $\mathscr{C}$. For two objects $X, X' \in \mathrm{Obj}(\mathscr{C})$, define

$$\delta^{\mathscr{F}}(X, X') := \inf \left\{ w(D) \;\middle|\; \begin{array}{l} D \text{ is an iterated cone decomposition} \\ \text{of } X \text{ (in } \mathscr{C}\text{) with a linearization as} \\ \text{with } \ell(D) = (F_0, F_1, ..., X', ..., F_k) \\ \text{with the objects } F_i \in \mathscr{F},\ k \in \mathbb{N}. \end{array} \right\}.$$

Moreover, define $d^{\mathscr{F}}(X, X') := \max\{\delta^{\mathscr{F}}(X, X'), \delta^{\mathscr{F}}(X', X)\}$.

Algebraic fragmentation pseudo-metric

Assume we can define $w(D) \in \mathbb{R}_{\geq 0}$, **weight of a cone decomposition**. One can give the following definition.

Definition

Let $\mathscr{F} \subset \mathrm{Obj}(\mathscr{C})$ be a family of objects in $\mathscr{C}$. For two objects $X, X' \in \mathrm{Obj}(\mathscr{C})$, define

$$\delta^{\mathscr{F}}(X, X') := \inf \left\{ w(D) \;\middle|\; \begin{array}{l} D \text{ is an iterated cone decomposition} \\ \text{of } X \text{ (in } \mathscr{C}\text{) with a linearization as} \\ \text{with } \ell(D) = (F_0, F_1, ..., X', ..., F_k) \\ \text{with the objects } F_i \in \mathscr{F},\ k \in \mathbb{N}. \end{array} \right\}.$$

Moreover, define $d^{\mathscr{F}}(X, X') := \max\{\delta^{\mathscr{F}}(X, X'), \delta^{\mathscr{F}}(X', X)\}$.

We will call $d^{\mathscr{F}}$ an **algebraic fragmentation pseudo-metric** if $d^{\mathscr{F}}$ satisfies the following triangle inequality,

$$d^{\mathscr{F}}(X, X') \leq d^{\mathscr{F}}(X, X'') + d^{\mathscr{F}}(X'', X'). \qquad (1)$$

Use topology to study algebra

Question

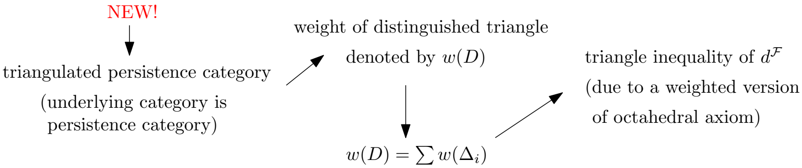
How to define a (non-trivial) $w(D)$?

Use topology to study algebra

Question

How to define a (non-trivial) $w(D)$?

One answer is from a joint work with P. Biran and O. Cornea (in progress). The main idea goes as follows.



Corollary

Let $\mathscr{C}$ be a triangulated persistence category, then $(\mathrm{Obj}(\mathscr{C}), d^{\mathscr{F}})$ is a topological space, and moreover, it is an H-space.

Use topology to study algebra

Question

How to define a (non-trivial) $w(D)$?

One answer is from a joint work with P. Biran and O. Cornea (in progress). The main idea goes as follows.

NEW!

triangulated persistence category (underlying category is persistence category)

weight of distinguished triangle denoted by $w(D)$

$w(D) = \sum w(\Delta_i)$

triangle inequality of $d^{\mathcal{F}}$ (due to a weighted version of octahedral axiom)

Corollary

Let $\mathscr{C}$ be a triangulated persistence category, then $(\mathrm{Obj}(\mathscr{C}), d^{\mathscr{F}})$ is a topological space, and moreover, it is an H-space.

Example (of a triangulated persistence category)

$\mathrm{Ho}_0(\mathscr{C})$ *where $\mathscr{C}$ is a* **filtered pre-triangulated** *dg-category.*

$\mathbf{k}$-linear category and categorical bimodule

- Recall that a $\mathbf{k}$-linear category $\mathscr{C}$ is a category where any hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a $\mathbf{k}$-module. (For instance, $\mathrm{Hot}^b(\mathscr{A})$ earlier is a $\mathbf{k}$-linear category.)

$\mathbf{k}$-linear category and categorical bimodule

- Recall that a $\mathbf{k}$-linear category $\mathscr{C}$ is a category where any hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a $\mathbf{k}$-module. (For instance, $\mathrm{Hot}^b(\mathscr{A})$ earlier is a $\mathbf{k}$-linear category.)

- A $\mathscr{C}$-bimodule is a functor

$$M : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k}),$$

where recall that $\mathrm{Mod}(\mathbf{k})$ is the category of $\mathbf{k}$-module.

$\mathbf{k}$-linear category and categorical bimodule

- Recall that a $\mathbf{k}$-linear category $\mathscr{C}$ is a category where any hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a $\mathbf{k}$-module. (For instance, $\mathrm{Hot}^b(\mathscr{A})$ earlier is a $\mathbf{k}$-linear category.)

- A $\mathscr{C}$-bimodule is a functor

$$M : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k}),$$

where recall that $\mathrm{Mod}(\mathbf{k})$ is the category of $\mathbf{k}$-module.

Decipher the definition.

(1) For $(X, Y) \in \mathrm{Obj}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})$, $M(X, Y)$ is a $\mathbf{k}$-module.

$\mathbf{k}$-linear category and categorical bimodule

- Recall that a $\mathbf{k}$-linear category $\mathscr{C}$ is a category where any hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a $\mathbf{k}$-module. (For instance, $\mathrm{Hot}^b(\mathscr{A})$ earlier is a $\mathbf{k}$-linear category.)

- A $\mathscr{C}$-bimodule is a functor

$$M : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k}),$$

where recall that $\mathrm{Mod}(\mathbf{k})$ is the category of $\mathbf{k}$-module.

Decipher the definition.

(1) For $(X, Y) \in \mathrm{Obj}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})$, $M(X, Y)$ is a $\mathbf{k}$-module.

(2) For $(f, g) \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}}((X, Y), (X', Y')) = \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, X') \otimes \mathrm{Hom}_{\mathscr{C}}(Y, Y')$,

$$M(f, g) \in \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k})}(M(X, Y), M(X', Y')),$$

i.e., for any $x \in M(X, Y)$, $M((f, g))(x) \in M(X', Y')$.

k-linear category and categorical bimodule

- Recall that a **k**-linear category $\mathscr{C}$ is a category where any hom-set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is a **k**-module. (For instance, $\mathrm{Hot}^b(\mathscr{A})$ earlier is a **k**-linear category.)

- A $\mathscr{C}$-bimodule is a functor

$$M : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k}),$$

where recall that $\mathrm{Mod}(\mathbf{k})$ is the category of **k**-module.

Decipher the definition.

(1) For $(X, Y) \in \mathrm{Obj}(\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C})$, $M(X, Y)$ is a **k**-module.

(2) For $(f, g) \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}} \otimes \mathscr{C}}((X, Y), (X', Y')) = \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, X') \otimes \mathrm{Hom}_{\mathscr{C}}(Y, Y')$,

$$M(f, g) \in \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k})}(M(X, Y), M(X', Y')),$$

i.e., for any $x \in M(X, Y)$, $M((f, g))(x) \in M(X', Y')$. In classical format,

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, X') \otimes M(X, Y) \otimes \mathrm{Hom}_{\mathscr{C}}(Y, Y') \to M(X', Y')$$

defined by $(f, x, g) \mapsto M(f, g)(x) =: f \cdot x \cdot g$.

Hochschild chain complex

Observe that every **k**-linear category $\mathscr{C}$ is a $\mathscr{C}$-bimodule, that is, $\mathscr{C} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k})$ by $\mathscr{C}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

Hochschild chain complex

Observe that every **k**-linear category $\mathscr{C}$ is a $\mathscr{C}$-bimodule, that is, $\mathscr{C} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k})$ by $\mathscr{C}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

- (complex of $\mathscr{C}$-bimodules) For each $n \geq 0$, define

$$C_n(\mathscr{C}, M) := \prod_{X_0,\ldots,X_n} M(X_n, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(X_{n-1}, X_n).$$

Note that $C_n(\mathscr{C}, M)$ is a **k**-module!

Hochschild chain complex

Observe that every **k**-linear category $\mathscr{C}$ is a $\mathscr{C}$-bimodule, that is, $\mathscr{C} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k})$ by $\mathscr{C}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

- (complex of $\mathscr{C}$-bimodules) For each $n \geq 0$, define

$$C_n(\mathscr{C}, M) := \prod_{X_0,\dots,X_n} M(X_n, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(X_{n-1}, X_n).$$

Note that $C_n(\mathscr{C}, M)$ is a **k**-module!

Remark

When $n = 0$, $C_0(\mathscr{C}, M) = \prod_{X \in \mathrm{Obj}(\mathscr{C})} M(X, X)$.

Hochschild chain complex

Observe that every **k**-linear category $\mathscr{C}$ is a $\mathscr{C}$-bimodule, that is, $\mathscr{C}:\mathscr{C}^{\mathrm{op}}\otimes\mathscr{C}\to\mathrm{Mod}(\mathbf{k})$ by $\mathscr{C}(X,Y)=\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

- (complex of $\mathscr{C}$-bimodules) For each $n\geq 0$, define

$$C_n(\mathscr{C},M):=\prod_{X_0,\ldots,X_n} M(X_n,X_0)\otimes\mathrm{Hom}_{\mathscr{C}}(X_0,X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{C}}(X_{n-1},X_n).$$

Note that $C_n(\mathscr{C},M)$ is a **k**-module!

Remark

When $n=0$, $C_0(\mathscr{C},M)=\prod_{X\in\mathrm{Obj}(\mathscr{C})}M(X,X)$.

Remark

From the complex above, one should have a feeling that Hochschild homology is related to cyclic homology.

Hochschild chain complex

Observe that every **k**-linear category $\mathscr{C}$ is a $\mathscr{C}$-bimodule, that is, $\mathscr{C} : \mathscr{C}^{\mathrm{op}} \otimes \mathscr{C} \to \mathrm{Mod}(\mathbf{k})$ by $\mathscr{C}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

- (complex of $\mathscr{C}$-bimodules) For each $n \geq 0$, define

$$C_n(\mathscr{C}, M) := \prod_{X_0,\ldots,X_n} M(X_n, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(X_{n-1}, X_n).$$

Note that $C_n(\mathscr{C}, M)$ is a **k**-module!

Remark

When $n = 0$, $C_0(\mathscr{C}, M) = \prod_{X \in \mathrm{Obj}(\mathscr{C})} M(X, X)$.

Remark

From the complex above, one should have a feeling that Hochschild homology is related to cyclic homology.

Consider the following complex (of **k**-modules),

$$\cdots \xrightarrow{\partial_3} C_2(\mathscr{C}, M) \xrightarrow{\partial_2} C_1(\mathscr{C}, M) \xrightarrow{\partial_1} C_0(\mathscr{C}, M) \to 0.$$

Hochschild boundary operator

The boundary operator ∂_n is defined as follows,

$$\partial_n(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{array}{l} ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ +\sum_{i=1}^{n-1} m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ +a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{array} .$$

Hochschild boundary operator

The boundary operator ∂_n is defined as follows,

$$\partial_n(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{array}{l} ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ +\sum_{i=1}^{n-1} m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ +a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{array} .$$

For a general element $m \otimes a_1 \otimes \cdots \otimes a_n$, there are three "compositions" involved in ∂_n.

- ma_1: $M(X_n, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_1) \to M(X_n, X_1)$.
- $a_i a_{i+1}$: $\mathrm{Hom}_{\mathscr{C}}(X_{i-1}, X_i) \otimes \mathrm{Hom}_{\mathscr{C}}(X_i, X_{i+1}) \to \mathrm{Hom}_{\mathscr{C}}(X_{i-1}, X_{i+1})$.
- $a_n m$: $\mathrm{Hom}_{\mathscr{C}}(X_{n-1}, X_n) \otimes M(X_n, X_0) \to M(X_{n-1}, X_0)$.

Therefore, $\partial_n : C_n(\mathscr{C}, M) \to C_{n-1}(\mathscr{C}, M)$.

Hochschild boundary operator

The boundary operator ∂_n is defined as follows,

$$\partial_n(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{array}{l} ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ + \sum_{i=1}^{n-1} m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{array} .$$

For a general element $m \otimes a_1 \otimes \cdots \otimes a_n$, there are three "compositions" involved in ∂_n.

- ma_1: $M(X_n, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_1) \to M(X_n, X_1)$.
- $a_i a_{i+1}$: $\mathrm{Hom}_{\mathscr{C}}(X_{i-1}, X_i) \otimes \mathrm{Hom}_{\mathscr{C}}(X_i, X_{i+1}) \to \mathrm{Hom}_{\mathscr{C}}(X_{i-1}, X_{i+1})$.
- $a_n m$: $\mathrm{Hom}_{\mathscr{C}}(X_{n-1}, X_n) \otimes M(X_n, X_0) \to M(X_{n-1}, X_0)$.

Therefore, $\partial_n : C_n(\mathscr{C}, M) \to C_{n-1}(\mathscr{C}, M)$.

Exercise

For any $n \geq 1$, $\partial_n \circ \partial_{n+1} = 0$.

Hochschild homology of a $\mathbf{k}$-linear category

Definition

Let $\mathscr{C}$ be a $\mathbf{k}$-linear category and M be a $\mathscr{C}$-bimodule. Then the **Hochschild homology of $\mathscr{C}$ with coefficient in M** is defined by

$$\mathrm{HH}_*(\mathscr{C}, M) := H_*(C_\bullet(\mathscr{C}, M), \partial_\bullet).$$

If $M = \mathscr{C}$, then for brevity, denote $\mathrm{HH}_*(\mathscr{C}) := \mathrm{HH}_*(\mathscr{C}, \mathscr{C})$.

Definition

Let $\mathscr{C}$ be a **k**-linear category and M be a $\mathscr{C}$-bimodule. Then the **Hochschild homology of $\mathscr{C}$ with coefficient in** M is defined by

$$\mathrm{HH}_*(\mathscr{C}, M) := H_*(C_\bullet(\mathscr{C}, M), \partial_\bullet).$$

If $M = \mathscr{C}$, then for brevity, denote $\mathrm{HH}_*(\mathscr{C}) := \mathrm{HH}_*(\mathscr{C}, \mathscr{C})$.

Remark

(1) Adding grading in the **k**-linear category $\mathscr{C}$ leads to a double complex and spectral sequence. (2) Hochschild **co**homology of $\mathscr{C}$ with coefficient in M is defined via $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}} \times \mathscr{C}}(\cdot, M)$.

Hochschild homology of a $\mathbf{k}$-linear category

Definition

Let $\mathscr{C}$ be a $\mathbf{k}$-linear category and M be a $\mathscr{C}$-bimodule. Then the **Hochschild homology of $\mathscr{C}$ with coefficient in M** is defined by

$$\mathrm{HH}_*(\mathscr{C}, M) := H_*(C_\bullet(\mathscr{C}, M), \partial_\bullet).$$

If $M = \mathscr{C}$, then for brevity, denote $\mathrm{HH}_*(\mathscr{C}) := \mathrm{HH}_*(\mathscr{C}, \mathscr{C})$.

Remark

(1) Adding grading in the $\mathbf{k}$-linear category $\mathscr{C}$ leads to a double complex and spectral sequence. (2) Hochschild **co**homology of $\mathscr{C}$ with coefficient in M is defined via $\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}} \times \mathscr{C}}(\cdot, M)$.

Example

If $\mathscr{C}$ has only one object, then $\mathscr{C}$ is a $\mathbf{k}$-algebra denoted by A, and a $\mathscr{C}$-bimodule is an A-bimodule. Moreover, $\mathrm{HH}_*(\mathscr{C}, M) = \mathrm{HH}_*(A, M)$.

Low degrees computations

Example (Continued from the previous example)

Recall that $\partial_1(m \otimes a) = ma + am = ma - am$. *So,* $\mathrm{im}(\partial_1) = [M, A]$. *Therefore,*

$$\mathrm{HH}_0(A, M) = M/[M, A].$$

In particular, if $M = A$, *then* $\mathrm{HH}_0(A) = A/[A, A]$. *Moreover, if* A *is commutative over* $\mathbf{k}$, *then* $\mathrm{HH}_0(A) = A$.

Low degrees computations

Example (Continued from the previous example)

Recall that $\partial_1(m \otimes a) = ma + am = ma - am$. *So,* $\mathrm{im}(\partial_1) = [M, A]$. *Therefore,*

$$\mathrm{HH}_0(A, M) = M/[M, A].$$

In particular, if $M = A$, *then* $\mathrm{HH}_0(A) = A/[A, A]$. *Moreover, if* A *is commutative over* $\mathbf{k}$, *then* $\mathrm{HH}_0(A) = A$.

Example

If A *is commutative over* $\mathbf{k}$, *then*

$$\mathrm{HH}_1(A, M) = M \otimes_A \Omega_{A/\mathbf{k}}$$

where $\Omega_{A/\mathbf{k}}$ *is the (degree-1) Kähler differential of* A.

Low degrees computations

Example (Continued from the previous example)

Recall that $\partial_1(m \otimes a) = ma + am = ma - am$. *So,* $\mathrm{im}(\partial_1) = [M, A]$. *Therefore,*

$$\mathrm{HH}_0(A, M) = M/[M, A].$$

In particular, if $M = A$, *then* $\mathrm{HH}_0(A) = A/[A, A]$. *Moreover, if* A *is commutative over* $\mathbf{k}$, *then* $\mathrm{HH}_0(A) = A$.

Example

If A *is commutative over* $\mathbf{k}$, *then*

$$\mathrm{HH}_1(A, M) = M \otimes_A \Omega_{A/\mathbf{k}}$$

where $\Omega_{A/\mathbf{k}}$ *is the (degree-1) Kähler differential of* A.

Exercise

$$\mathrm{HH}_i(\mathbf{k}) = \begin{cases} \mathbf{k} & \text{if} \quad i = 0 \\ 0 & \text{if} \quad i \geq 1 \end{cases}.$$

Why do we care about HH_*?

Conjecture (Kontsevich 1994 ICM)

Let M be a compact symplectic manifold. Then

$$\mathrm{QH}^*(M) \simeq \mathrm{HH}^*(\mathrm{Fuk}(M))$$

Why do we care about HH_*?

Conjecture (Kontsevich 1994 ICM)

Let M be a compact symplectic manifold. Then

$$\mathrm{QH}^*(M) \simeq \mathrm{HH}^*(\mathrm{Fuk}(M))$$

Hence, one recovers quantum ring structure from computation of the Fukaya category. Under certain conditions, this conjecture is verified by Ganatra in preprint: *Automatically generating Fukaya categories and computing quantum cohomology (2016-2019)*.

Why do we care about HH_*?

Conjecture (Kontsevich 1994 ICM)

Let M be a compact symplectic manifold. Then

$$\mathrm{QH}^*(M) \simeq \mathrm{HH}^*(\mathrm{Fuk}(M))$$

Hence, one recovers quantum ring structure from computation of the Fukaya category. Under certain conditions, this conjecture is verified by Ganatra in preprint: *Automatically generating Fukaya categories and computing quantum cohomology (2016-2019)*.

Theorem (Seidel 2002 ICM, Ganatra 2012)

Let M be a Liouville manifold, and denote by $\mathrm{W}(M)$ the wrapped Fukaya category of M. Then, under a certain condition, one has

$$\mathrm{SH}^*(M) \simeq \mathrm{HH}^*(\mathrm{W}(M)).$$

Another important progress: Abouzaid's generating criterion (Publ. IHES 2010).

Desired map f_2 (character)

Proposition (cf. Proposition 3.8 in Seidel's book)

Let $\mathscr{C}$ be an enhanced triangulated **k***-linear category (i.e., a homotopy category of a pre-triangulated* **k***-linear dg-category). Then there exists a well-defined homomorphism*

$$f_2 : K_0(\mathscr{C}) \to \mathrm{HH}_0(\mathscr{C}) \quad \textit{by} \quad f_2([X]) = [e_X]$$

for any $X \in \mathrm{Obj}(\mathscr{C})$, where $e_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ is the identity map.

Note that $e_X \in \mathrm{Hom}_{\mathscr{C}}(X, X) \subset C_0(\mathscr{C}, \mathscr{C})$.

Desired map f_2 (character)

Proposition (cf. Proposition 3.8 in Seidel's book)

Let $\mathscr{C}$ be an enhanced triangulated $\mathbf{k}$-linear category (i.e., a homotopy category of a pre-triangulated $\mathbf{k}$-linear dg-category). Then there exists a well-defined homomorphism

$$f_2 : K_0(\mathscr{C}) \to \mathrm{HH}_0(\mathscr{C}) \quad \text{by} \quad f_2([X]) = [e_X]$$

for any $X \in \mathrm{Obj}(\mathscr{C})$, where $e_X \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ is the identity map.

Note that $e_X \in \mathrm{Hom}_{\mathscr{C}}(X, X) \subset C_0(\mathscr{C}, \mathscr{C})$.

Example

Consider $\mathscr{C} = \mathrm{Hot}^b(\mathrm{Mod}(\mathbf{k}))$. Recall that $K_0(\mathscr{C}) = K_0(\mathrm{Mod}(\mathbf{k}))$. If $[X] = [A] + [B]$ in $K_0(\mathrm{Mod}(\mathbf{k}))$, then $X \simeq A \oplus B$. Then

$$e_X = e_{A \oplus B} = e_A + e_B$$

Then $f_2([X]) = [e_X] = [e_A] + [e_B] = f_2([A]) + f_2([B]) = f_2([A] + [B])$.