Higher symplectic capacities (ref: Kyler Siegel)

Tuesday, June 30, 2020 7:52 PM

Last time: • X Liouville domain

S.E.S

$$0 \to C_*^{S^1}(X,\partial X) \to SC_*^{S^1}(X) \to SC_*^{S^1,+}(X) \to 0$$

L.E.S

$$SH_*^{S^1,+}(X) \xrightarrow{\delta[-1]} H_*^{S^1}(X,\partial X) = H_*(X,\partial X)\otimes$$
$$= H_*(X,\partial X)\otimes k$$
$$SH_*^{S^1}(X)$$

when $SH_*^{S^1}(X)=0$ (e.g. X Ellipsoid), then $Im(\delta[-1]) = H_*$

define $c_1(X) = \inf\left\{ a \;\middle|\; \begin{array}{l} \delta(x) = e = [X,\partial X]\otimes 1 \\ \text{for some } x \in SH^{S^1,+,\le a}(X) \end{array} \right\}$

$c_k(X) = \inf\left\{ a \;\middle|\; \begin{array}{l} \delta U^{k-1}(x) = e \text{ for some} \\ x \in SH^{S^1,+,\le a}(X) \end{array} \right\}$

$= \inf\left\{ a \;\middle|\; \begin{array}{l} U^{k-1}\delta(x) = e \quad \cdot \quad \cdots \\ \cdot \quad - \quad - \quad - \end{array} \right\}$

$= \inf\left\{ a \;\middle|\; \begin{array}{l} \boxed{\delta}(x) = e\otimes u^{k-1} + \text{l.o.t. w.r.t. } u \text{ for} \\ \text{some } x \in SH^{S^1,+,\le a}(X) \end{array} \right\}$

<u>Today</u>: Using the L_∞ structure on $SC^{S^1,+}$

and an extension of δ to an L_∞ homomor

(roughly) $\hat{\delta} : \bigoplus_{k=1}^{\infty} Sym^k(SC_*^{S^1,+}) \longrightarrow \bigoplus_{k=1}^{\infty} Sym^k(C_*^{S^1}(X,\partial X))$

New capacities:

$\inf\{\sigma \mid$ certain elts being hit by $\delta\}$.

Plan: ① L_∞ alg
② L_∞ structure on $SC^{s!,t}$
③ L_∞ homomorphism $\hat{\delta}$
④ Example

§1 L_∞ algebra

- $\mathbb{K}$ coeff ring contains $\mathbb{Q}$
- V graded $\mathbb{K}$-module (e.g. $V = SC_{**}$.)
- $\odot^k V = V^{\otimes k} / S_k = Sym^k(V)$ (unordered k. of orbits
- $\bar{S}V = \bigoplus_{k=1}^{\infty} \odot^k V$ (put all tuples of ord together
- $sh(i,k-i) = \left\{ \sigma \in S_k \;\middle|\; \begin{array}{l} \sigma(1) < \cdots \sigma(i) \\ \sigma(i+1) < \cdots \sigma(k) \end{array} \right\}$

(partitions of k elements into 2 sets $\{\sigma(1), \cdots \sigma(i)\}$ $\{\sigma(i+1), \cdots \sigma(k)$

- $\Delta : \bar{S}V \longrightarrow \bar{S}V \otimes \bar{S}V$

$$\Delta(v_1 \odot \cdots \odot v_k) = \sum_i \sum_{\sigma \in sh(i,k-i)} v_{\sigma(1)} \odot \cdots v_{\sigma(i)} \otimes v_{\sigma(i+1)} \odot$$

(sum of all possible ways to split $v_1, \cdots v_k$ into two)

<u>Def</u>: An L_∞ algebra consists of

a) V graded $\mathbb{K}$-module

b) $\hat{l} : \bar{S}V \to \bar{S}V$ s.t.

$$\begin{cases} \Delta \circ \hat{l} = (1 \otimes \hat{l}) \circ \Delta + (\hat{l} \otimes 1) \circ \Delta & \text{— ①} \\ \hat{l}^2 = 0 & \text{— ②} \end{cases}$$

(grading & signs suppressed

In practice, $l^k := \pi_1 \circ \hat{l} : \odot^k V \longrightarrow V$ multilinear for $k = 1, \ldots$

Conversely, given a seq. multilinear maps

$$l^k : V^{\otimes k} \longrightarrow V$$

it uniquely determines

$$\hat{l} : \bar{S}V \longrightarrow \bar{S}V \quad \text{s.t. ① is sa}$$

$$\hat{l} = \sum_k \sum_{\sigma \in S_n} \frac{1}{k!(n-k)!} l_k(v_{\sigma(1)}, \ldots v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots$$

$$\pi_1 \circ \hat{l} : \odot^k V \longrightarrow V = \text{symmetrization } l^k$$

① + ② repackage L_∞-relations among $\{l^k\}$

<u>Example</u>: ① $\hat{l} \circ \hat{l}(v) = 0$

$$\Rightarrow l^1 \circ l^1(v) = 0 \quad \text{(differential)}$$

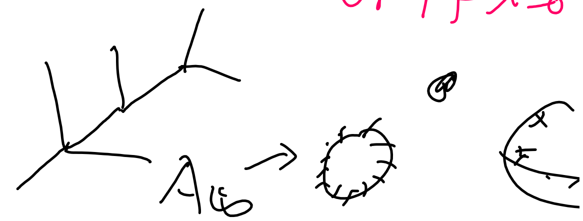
② $\hat{l} \circ \hat{l}(v_1, v_2) = 0$

$$\Rightarrow l^2(l^1(v_1), v_2) + l^2(v_1, l^1(v_2)) + l^1(l^2(v_1, v_2))$$

(derivation)

③ $\hat{l} \circ \hat{l}(v_1, v_2, v_3) = 0 \Rightarrow$ (Jacobi identity if $l^3 = 0$

or if $l'=0$



Def: L_∞ homomorphism from $(V, \hat{l}_V)$ to $(W, \hat{l}_W)$

$$\hat{\Phi} : \bar{S}V \longrightarrow \bar{S}W \text{ s.t. } \begin{cases} \Delta \circ \hat{\Phi} = (\hat{\Phi} \otimes \hat{\Phi}) \circ \Delta \\ \hat{l}_W \circ \hat{\Phi} = \hat{\Phi} \circ \hat{l}_V \end{cases}$$

$\hat{\Phi} \rightsquigarrow \{\Phi^k\}$

It is called a quasi-isom if

~~$\Phi^1 : (V, l^1_V) \longrightarrow (W, l^1_W)$ chain~~

- $\Phi^1 : H(V, l^1_V) \xrightarrow{\cong} H(W, l^1_W)$

If $\mathbb{K}$ is a field, then it defines an equivalence relation for L_∞ algebra

Bourgeois–Oancea

§2 L_∞ structure on ~~$SH^{S^1,+}_*(X)$~~

linearized contact homology CH_{lin}

- Coeff ring $\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{c_i} \;\middle|\; a_i \in \mathbb{C},\ 0 \le c_0 < c_1 < \cdots,\ \lim_i \right.$

- $\mathcal{X}$ = set of good Reeb orbits on $(Y, \alpha) =$

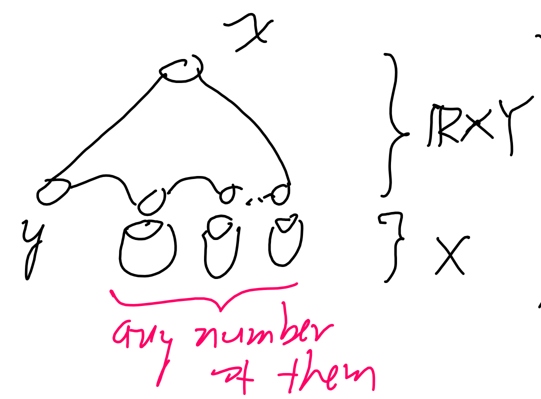
Detail: We choose a base point for each simple orbit, $\mathcal{X} = \cdots\cdots\cdots\ x$ s.t. $x(0) =$ base

(~~clear~~ base point $\longleftrightarrow$ S^1-quotient)

<u>Def</u>: The linearized contact chain complex CC_{lin}

- $\Lambda_0 X$ (as graded module)
- differential

$$\partial x = \sum_{\substack{y \in X \\ A \in H_2(X, x \cup y)}} \#\Big($$

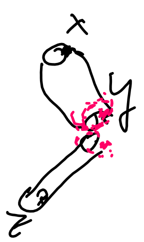


$$\Big)_A \frac{T^{\omega \cdot A}}{\kappa_y}\, y$$

Kill amb

where κ_y = multiplicity of y @

$\partial^2 = 0$

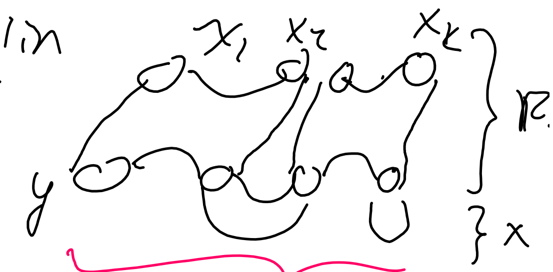


$\longrightarrow$

L_∞ structure

$$l^k : CC_{lin}^{\otimes k} \longrightarrow CC_{lin}$$

$$l^k(x_1, \dots, x_k) = \sum_{\substack{y \in X \\ A \in H_2(X, x_1 \cup \dots \cup x_k \cup y)}} \#\Big($$



<u>Claim</u>: $\{l^k\}$ satisfy L_∞ relations

$\{\}$ $\rightsquigarrow$ L_∞-homomorphism $\bar{S} CC_{lin} \longrightarrow \bar{S} C$

Claim: L_∞-structure on $C_*^{S^1}(X,\partial X)$ is trivial

$(C_*^{S^1}(X,\partial X), \hat{l}_{C_*^{S^1}}) \underset{\text{q-isom}}{\simeq} (\boxed{H_*^{S^1}(X,\partial X), \hat{l}_{H_*^{S^1}} = 0}$

after inverting T

$SH^{S^1,+}$

Want: $\hat{\delta}: \bar{S}CC_{1,n}^* \longrightarrow \bar{S}(H_*^{S^1}(X,\partial X))$

$\boxed{\hat{\delta} \circ \hat{l}_{CC_{1,n}} = \cancel{\hat{l}_{H_*^{S^1}}}^{0} \circ \hat{\delta} = 0}$

SH

To understand $\hat{\delta}$, it suffices to understand

$\delta_m^k(-) \overset{\text{def}}{=} \langle \underbrace{\delta^k(-)}_{H_*^{S^1}(X,\partial X)}, u^m \rangle \quad \forall k, m$

$\in H_*(X,\partial X) = \Lambda$

and if X = ellipsoid

$\Rightarrow H_*^{S^1}(X,\partial X) = \boxed{H_*(X,\partial X)} \otimes \Lambda_0[u] = \Lambda_0[u]$

$\boxed{H_*(X,\partial X)} \to \Lambda_0$

$\boxed{\hat{\delta}_m} : \bar{S}CC_{1,n} \longrightarrow \bar{S}\Lambda_0$

$\hat{\delta} \circ \hat{l}_{CC_{1,n}} = 0 \Rightarrow \hat{\delta}_m \circ \hat{l}_{CC_{1,n}} = 0 \quad \forall m$

Now: $\forall m \in \mathbb{N}_{\geq 0}$, $\hat{\varepsilon}_m : \bar{S}CC_{1,n} \longrightarrow \bar{S}\Lambda_0$

(but relation between $\hat{\varepsilon}_m$ & $\hat{\delta}_m$ is not transparent

$\underline{m=0}$: $\hat{\varepsilon}_m$, ε_m^k

$x_1 \; x_2 \; x_3 \quad x_k$

$\varepsilon_m^k(x_1, \dots x_k) = \sum \#\left(\quad \right)_A T^{w_A}$

$\hat{\delta}$

claim: $\hat{\mathcal{E}}_m \circ \hat{\ell}_{CC_{lin}} = 0$ for $m=0$

$m \geq 1$:

$$\mathcal{E}^k_m(x_1, \dots, x_k) = \sum_{A \in H_2(X, x_1 \cup \dots \cup x_k)} \#\Big(\dots \Big)_A T^{\dots}$$

$x_1 \ x_2 \ x_3 \ x_k$

T^m_p $u^m e$

(mark pt $\longmapsto$ $p \in X$ w/ one tangency w/ a divisor D_p at

claim: $\hat{\mathcal{E}}_m \circ \hat{\ell} = 0 \quad \forall m$

Define: $\mathcal{E}^k(-) := \sum_{m=0}^{\infty} \mathcal{E}^k_m(-) u^m \in \Lambda_0[[u]]$

$\rightsquigarrow \hat{\mathcal{E}} : \bar{S} CC_{lin} \to \bar{S} \Lambda_0[[u]]$

Capacities: For $b \in \bar{S}\mathbb{C}[[u]]$ and $w \in \mathbb{N}_{\geq 0}$

$$g_b^{\leq w}(X) = \inf \left\{ a \;\middle|\; \begin{array}{l} T^a b = \hat{\mathcal{E}}(x) \text{ for some} \\ x \in \bar{S}_{\leq w} CC_{lin}(X) \text{ s.t. } \hat{\ell}(x \dots \end{array} \right.$$

$\bar{S}_{\leq w} CC_{lin}(X) = \bigoplus_{k=1}^{w} \odot^k CC_{lin}(X)$

e.g. $g^{\leq 1}_{u^m}(X) = \inf \left\{ a \;\middle|\; \begin{array}{l} T^a = \mathcal{E}^1_m(x) \text{ for some} \\ x \in CC_{lin} \text{ w/ } \ell^1(x) = 0 \end{array} \right\}$

(analogous to Gutt-Hutchings capacities

(... $\hat{\mathcal{E}}$, $\hat{\ell}$ are related?)

Example $X = D^2$

Last time

$$SC^{S^1}_*(D^2) = \partial^{S^1} = \partial + u\varphi_1 + u^2\varphi_2 + \cdots$$

orbits	e	$\check{x}_1$	$\hat{x}_1$	$\check{x}_2$	$\hat{x}_2$	$\dots$
Action	0	1	1	2	2	$\dots$
Deg	1	2	3	4	5	$\dots$

$\varphi_k = 0 \ \forall k \geq 2$, $\varphi_1(\check{x}_\ell) = \ell \hat{x}_\ell$ for $\ell = 1, 2, \dots$

$SH^{S^1,+}_*(D^2)$ is generated by

$$\check{x}_k - \frac{Tu}{k-1}\check{x}_{k-1} + \cdots + (-1)^k \frac{T^{k-1}u^{k-1}}{(k-1)!}\check{x}_1 \quad \text{for}$$

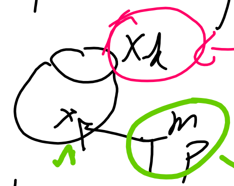
w/ δ-image $= (-1)^{k-1}\frac{T^{k-1}u^{k-1}}{(k-1)!}$

$CC_{lin}(D^2)$ is generated by

orbits	x_1	x_2	x_3
Action	1	2	3
Deg	2	4	6

$\ell^k = 0 \ \forall k$ for degree reason

$\varepsilon^1_m(x_k) = \sum_A \#\Big($



$\Big)_A \, T^{\omega \cdot A}$

$$= \#\begin{pmatrix} \mathbb{P}^1 \to \mathbb{P}^1 \text{ w/ branch pt} \\ \text{at } p \text{ of order } m+1 \\ \text{at } \infty \text{ of order } k \end{pmatrix} T^{\deg(x_k)}$$

$$= \begin{cases} 1 \cdot T^k & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{E}^1_m$ characteristic function $\chi_{x_{m+1}}$

$$\Rightarrow \quad \hat{\mathcal{E}}: \quad \mathcal{E}^1: \mathcal{C}_{lin}(D^2)[T^{-1}] \xrightarrow{\wedge} \Lambda_0[u][T^{-1}]$$

$$x_k \longmapsto u^{k-1}$$

$$\begin{array}{ccc}
SC^{S^1,+}_{*}(D^2) & \xrightarrow{BO} & \mathcal{C}_{lin}(D^2) \\
\check{x}_k \cdots (-1)^{k-1}\dfrac{T^{k-1}u^{k-1}}{(k-1)!} & \longmapsto & x_k \\
\hat{x}_k & \longmapsto & 0 \\
u & \longmapsto & 0 \\
\check{x}_k & \longmapsto & x_k \\
\Big\downarrow \partial_{S^1} & \langle \delta^1, u^m \rangle & \Big\downarrow \mathcal{E}^1_m \\
(-1)^{k-1}\dfrac{(Tu)^{k-1}}{(k-1)!} & \xrightarrow{\langle u^m, - \rangle} & \begin{cases} \dfrac{T^{k-1}}{(k-1)!}(-1)^{k-1} \\ 0 \end{cases}
\end{array}$$

commute up to unit

$$\boxed{\langle \delta^1, u^m \rangle = (\text{unit}) \mathcal{E}^1_m}$$

$$\langle \delta^k, u^m \rangle \longleftrightarrow \mathcal{E}^k_m$$

$\varepsilon_{E,m}($

For D^2; $CC_{1m}[T^{-1}] \xrightarrow{\varepsilon} \Lambda_0[u][T^{-1}]$

For general $X \supseteq E(a, a', a', \cdots a') \ni p$

$$\underline{\hat{\varepsilon}_X} = \boxed{\hat{\varepsilon}_{E(a,\dots)}} \circ \underline{\hat{\Phi}} \leftarrow [$$

$\hat{\Phi} : \tilde{S}CC_{1m}(X) \to \tilde{S}CC_{1m}(E)$

$\hat{\varepsilon}_X \cancel{=} \hat{\varepsilon}_E$

ε'_m

For any $c > 0$, $m, k \in \mathbb{N}$, $\exists\, a << a'$

$$\boxed{\varepsilon^1_{E(a,a',\dots),m}(\alpha_k) = \varepsilon^1_{D^2,m}(\alpha_k)} \bmod$$

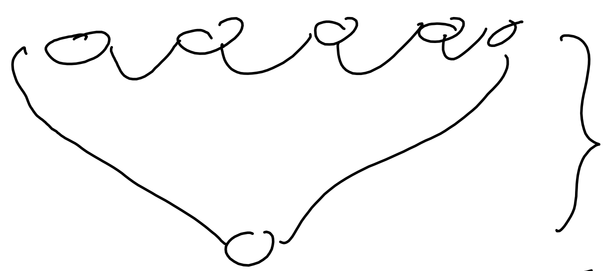
where α_k = k-fold iterate of shortest simple Reeb

For X Ellipsoid

$\hat{\varepsilon} : \tilde{S}CC_{1m}(X) \to \tilde{S}\Lambda_0[u]$ is a

after inverting

Example:



captured by L_∞ struc

$E(1, 13/2) \hookrightarrow E(c, c)$

Consider $E(1, 13/2 + \delta) \hookrightarrow E(c, c + \delta')$

$\beta_1' \quad \beta_1' \quad \beta_1' \quad \beta_1' \quad \beta_1'$

CG-Hind :  $\Big\}\ E(c,c+\delta')\setminus E(1,$

for c large

$\beta_1' =$ *longer* simple Reeb orbit of $\partial E(c,c+\delta')$

$\alpha_{13} =$ 13-fold cover of shorter simple Reeb orbit of $\partial E(1,\frac{13}{2}+\delta$

<u>Claim</u>: $\exists\, b \in \mathbb{C}[u]$ s.t.

$$\begin{cases} g_b(E(c,c+\delta')) = 5\mathcal{A}(\beta_1') = 5(c+\delta') \\ g_b(E(1,\frac{13}{2}+\delta)) \geq 13 = \mathcal{A}(\alpha_{13}) \end{cases} \Rightarrow 13 \leq 5(\Rightarrow \boxed{c \geq \frac{13}{}}$$

<u>Property</u>: $g_b(X) = g_b(X\times\mathbb{C})$

ECH $\alpha^k = \alpha^a \otimes \alpha^{k-a}$ CClin: $\alpha^k \neq \alpha^a \otimes$