Symplectic Cremona Maps

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Background: Cremona Maps in Birational Geometry

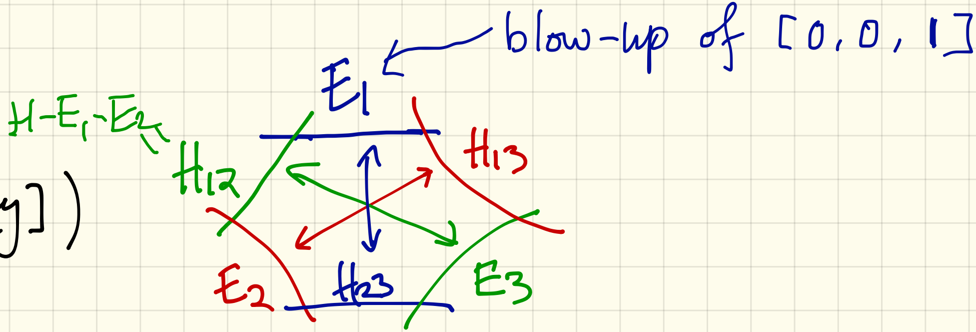
Def: A Cremona map (transform) is a birational automorphism of $\mathbb{P}^n_k$, k = any field.

We will restrict ourselves to $n=2$ & $k=\mathbb{C}$ only

A basic result: (M. Noether)

A plane Cremona map can be factorized into a composition of quadratic transforms.

$$([x,y,z] \mapsto [yz, xz, xy])$$



Further Questions: What are the finite subgroups?

$n=1$: F. Klein, Cyclic, dihedral, tetrahedral, octahedral, isocahedral

$n=2$: (I. Dolgachev, A. Iskovskikh, 2009)

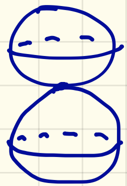
Some Preliminaries:

1) We can always focus on minimal G-actions

$+$ G-inv. disjoint set

$\Longleftrightarrow$ there is no G-invariant exceptional curves.

2) Conic bundles: A G-conic bundle structure on X means X is fibered by $\mathbb{P}^1$ with possibly finite singular fibers, each $\cong \mathbb{P}^1 \wedge \mathbb{P}^1 =$



, fiberation G-equivariant.

Del Pezzo: $-K_S$ is ample.

Another classical result (a cornerstone for Dolgachev–Iskovskikh's classification)

Theorem: Let S be a minimal rational G-surface, then either:

1) S admits a structure of conic G-bundle with $\mathrm{Pic}^G = \mathbb{Z}^2$

OR

2) S is isomorphic to a Del Pezzo.

Switching Gears to Symplectic Geometry:

Notions on Symplectic birational Geometry: (Hu-Li-Ruan, Li-Ruan)

In general, it is difficult to define a birational map on a point-to-point basis between symp. mfds (even for BU/BD), but:

Theorem: (Abramovich-Karu-Matsuki-Wlodarczyk) A birational map between projective mfds can be decomposed into a sequence of BU/BD's.

Inspired by this:

Def: Two symplectic mfds are birationally equivalent if they are related by a sequence of BU/BD's.

Natural Candidates in Symplectic Cremona Theory:

AG	SG
Birational maps on $\mathbb{P}^2$	$\mathrm{Symp}(\mathbb{P}^2 \# n\overline{\mathbb{P}}^2)$
Quadratic transforms on $\mathbb{P}^2$	Symplectic Dehn twists
Del Pezzo surfaces	Monotone symplectic rational surfaces ($c_1 = \kappa[\omega]$)

Both are topologically $\mathbb{P}^2 \# n\overline{\mathbb{P}}^2$, $n \leq 8$

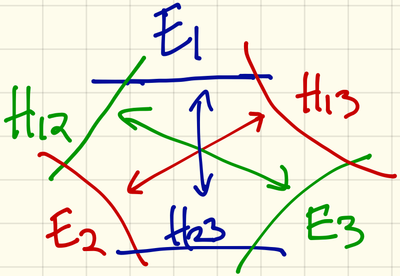
Symplectic Dehn twists: (P. Seidel)

Def: (Sketch) Use time-π unit geodesic flow on $T^*S^2 \setminus \{zero\}$
+ slow down near infinity + antipodal map at $\{zero\}$
+ Weinstein neighborhood theorem.

Main Properties: 1) Supported near a Lag. S^2: $(=L)$
2) Homological action: $A \longmapsto A + (A \cdot [L]) \cdot [L]$

Particular case:

$[L] = H - E_1 - E_2 - E_3$
in $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$



$\Rightarrow$ identical to that of quadratic transform. But has exotic symplectic behaviors ($\tau^2 \neq id$ in general)

Theorem: (Homological Factorization of symplectic Cremona Maps)
(Li–W. 12') $f \in Symp(\mathbb{P}^2 \# n\overline{\mathbb{P}}^2 = M_n)$, then $\exists L_1, \cdots, L_n \subset M_n$, Lag. spheres, s.t. $f_* = (\tau_{L_1})_* \cdots (\tau_{L_n})_*$ acting on $H_2(M_n)$.

Sketch of ideas: The key is to use symplectic areas.

1) Homological action determined by $f_*(E_i)$, $\{H, E_i\}$ a basis of H_2

2) Choose basis $\{E_i\}$ so that: $\begin{cases} \omega(E_1) = \min\limits_{\mathcal{E}} \omega(e) \\ \omega(E_k) = \min\limits_{e \in \mathcal{E}, (e, E_i) = 0, i \leq k-1} \omega(e) \end{cases}$

Let $f_*(E_i) = aH - \sum b_i E_{i_k}$, $b_i \geq b_{i+1}$

$\Longrightarrow f_*(E_i) \xrightarrow{R_{H_{123}}} \cdots \longrightarrow \cdots$

Reflection w.r.t. $H - E_1 - E_2 - E_3$

① Along this sequence $f_*(E_i) \to E_i$

② ω-area is monotone $\searrow$

$\Longrightarrow \omega(f_*(E_i)) = \omega(E_i) \Rightarrow$ all involved reflections are geometrically realized by τ_{L_i}.

Theorem: (Chen-Li-W.) G is a finite group acting on M_n symplecti

Assume $M_n = (\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2, \omega, J_0) =$ Kähler. Then either:

1) (X, J_0) is Del Pezzo, ω is monotone.

OR

2) X has a G-conic bundle structure.

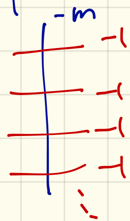
Note: 1) J_0 need not be G-invariant, but needed to apply Mori theory.

2) It is not known if such an integrable J_0 always exists, but true for $n \leq 8$ (Li-Zhang)

Sketch of proof:

1. Take a G-invariant J, then $\exists$ J-fibration of M_n.

Configuration analysis $\Rightarrow$ Singular fibers $=$



Problem: Want to conclude $\exists$ fiber class F, $G \cdot F = F$

unless X is Del Pezzo.

2. Assume $G = \mathbb{Z}/p\mathbb{Z}$ for simplicity

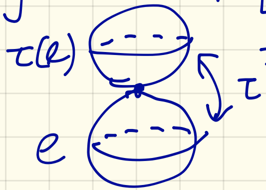
Take e s.t. $e \in \mathcal{E}$, $\omega(e) = \min \omega(\mathcal{E})$.

Recall: $\overline{NE}(X) = $ Cone of curves $= \{ \sum a_i [C_i] \mid C_i \subset (X, J_0), \text{ irred}, a_i \geq 0 \} / \sim$ numerical equivalence.

Lemma: If $Ge = Ga + Gb$, $a, b \in \overline{NE}(X)$, then $\mathbb{R}_+ Ge = \mathbb{R}_+ Ga = \mathbb{R}_+ Gb$

(Ge is G-extremal) $\quad Ge = \sum_{g \in G} g \cdot e$

$\boxed{\text{Case (I)}}$: $(Ge)^2 \le 0 \xrightarrow[\text{adjunction}]{} G = \mathbb{Z}_2$, genus $= 0 \Rightarrow [\tau(e) + e]$ is a fiber class



$\Rightarrow$ Conic bundle

$\boxed{\text{Case (II)}}$: $(Ge)^2 > 0$, then Ge has J_0-holomorphic Rep., so not hard to find ample line bdle $L \cdot (Ge) > 0$.

$\xrightarrow[\text{birational geom}]{\text{A result in}} Ge \subset \mathrm{Int}(\overline{NE}(X))$

$\xrightarrow{\text{Ge extremal}} \mathbb{R}_+ (G \cdot \overline{NE(X)}) = \mathbb{R}_+ (Ge)$

$\Longrightarrow K(X) \cdot \overline{NE(X)} < 0$

$\xrightarrow{\text{Kleiman}} K(X)$ is ample $\Rightarrow X$ Del Pezzo.

Further Questions & Developments

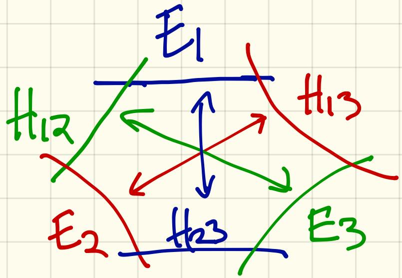
1. Very Recently we found a more symplectic proof which removes the restriction that ω needs to be a Kähler form for some J_0.

2. Complete classification of Cremona subgroups? Comparison with Dolgachev–Iskovskikh?

   1) The case of G-conic bundles descend to the base $\mathbb{P}^1$

   2) Del Pezzo case: $n \le 2$, easier to handle

      $n = 3$, focus on exceptionals



$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow H \longrightarrow 1$$

homologically trivial (G_0)   induced homological action (H) ✓

$n \ge 4$: Complicated configurations, many (but finite) exceptionals.