Dehn twist exact sequences through Lagrangian Cobordisms

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Background:

1) Seidel's exact sequence & its variants

(Seidel)

$$\begin{tikzcd} HF(S^n, L_1) \otimes HF(L_0, S^n) \arrow[r] & HF(L_0, L_1) \arrow[r] & HF(L_0, \tau_{S^n} L_1) \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$

(Seidel, fixed pt version)

$$\begin{tikzcd} HF^*(\tau \circ f) \arrow[r] & HF^*(f) \arrow[r] & HF(f(S^n), S^n) \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$

(Wehrheim-Woodward, family version

$$\begin{tikzcd} HF(C \circ L_0, C \circ L_1) \arrow[r] & HF(L_0, L_1) \arrow[r] & HF(L_0, \tau_C L_1) \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$

Lag. composition,

C = spherical coisotropic.

* We will provide new proofs of these results.

2) Lagrangian surgeries & cobordisms (Lalonde-Sikorav, Polterovich)
Biran - Cornea

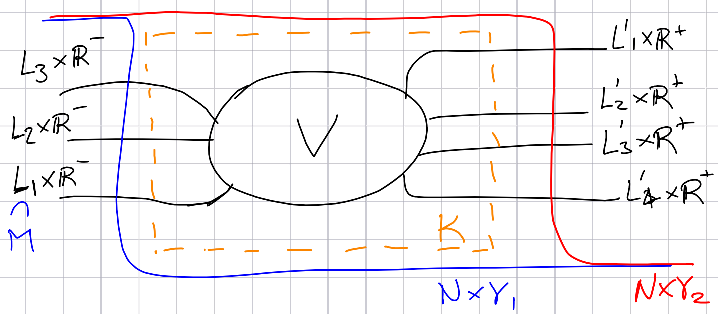
i) (L-S, P): When $L_1 \pitchfork L_2 = \{p\} \Rightarrow \exists\, L_1 \#_p L_2 \hookrightarrow M$

by adding handle in a Darboux chart $\mathcal{U} \ni p$.

Mak-W: Several approaches to "global surgeries", includes clean intersections.

ii) (B-C) Can construct Lag. cobordisms associated to surgery $L_1, L_2 \rightsquigarrow L_1 \#_p L_2$

One picture recap for Biran-Cornea:



$\subset M \times \mathbb{C} \Rightarrow$ $Cone(L_1 \to L_2 \cdots \to L_n) \simeq Cone(L_1' \to \cdots \to L_n')$

Mak-W. : Construction & Cobordism formalism works for

a) Clean surgeries b) immersed Lag.

3) Huybrechts-Thomas conjecture:

Seidel: Lag S^n $\xleftrightarrow{\text{mirror}}$ spherical object $\begin{cases} \mathrm{Hom}^*(\mathcal{E},\mathcal{E}) \simeq H^*(S^n) \\ \mathcal{E}\otimes\omega \simeq \mathcal{E} \end{cases}$

Dehn twist $\xleftrightarrow{\text{mirror}}$ $\mathcal{F} \longmapsto \mathrm{Cone}(\hom(\mathcal{E},\mathcal{F})\otimes\mathcal{E} \longrightarrow \mathcal{F})$

Huybrechts-Thomas:

Lag $\mathbb{CP}^n$ $\xleftrightarrow{\text{mirror}}$ projective object $\begin{cases} \mathrm{Hom}^*(\mathcal{E}_P,\mathcal{E}_P) \simeq H^*(\mathbb{CP}^n) \\ \mathcal{E}\otimes\omega \simeq \mathcal{E} \end{cases}$

Dehn twists $\xleftrightarrow{\text{mirror}}$ $\mathcal{F} \to \mathrm{Cone}(\hom(\mathcal{E},\mathcal{F})\otimes\mathcal{E}[-2] \to \hom(\mathcal{E},\mathcal{F})\otimes\mathcal{E}$
$\to \mathcal{F})$

Question: Really ?!?

Mak-W. : YES ! !

Side Remark: i) Spherical object & projective objects are adapted to (true) $SU(n)$ - CY & hyperkähler, resp.

all line bundles are s/p-objects, etc.

(Bogomolov) $\Rightarrow$ builds all CY with $\pi_1 = 0$

ii) For $L = S^2 = \mathbb{CP}^1$, $(\tau_{S^2})^2 = (\tau_{\mathbb{CP}^1})$. Part of how the projective twist formula comes.

iii) Works for other projective spaces.

iv) Will determine some connecting maps.

Basic Idea : A Mirror Proof

Spherical / Projective twist are defined by

Fourier–Mukai transforms, e.g. $P_{\mathcal{E}} = FM_{\mathrm{cone}(\mathcal{E} \boxtimes \mathcal{E} \to \Delta)}$

Philosophy: 1) Symplectic Fourier–Mukai $\approx$

Lagrangian composition $L_0 \xrightarrow{\circ L_{01}} L_1$

2) Cone in FuK $\approx$ Lagrangian cobordism / surgery

Upshot:

Cobordism/surgery in prods. $\xleftrightarrow{\text{Mirror}}$ Cones on FM-kernels.

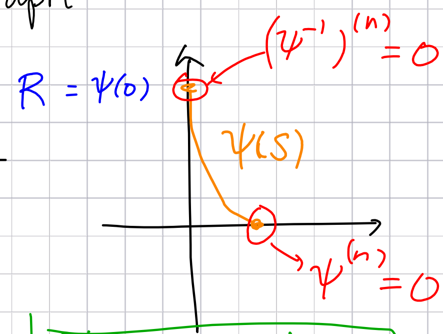
$\Downarrow$

Graph of Dehn twists

Surgeries Through geodesic flows

Set-up:

(1) Let $\psi(s)$ be a function whose graph "looks like"



(2) $D \subset L_1 \overset{\text{Lag}}{\subset} M$, N_D^* conormal bdle

(3) ϕ_t is the Ham. flow gen by $\|p\|$

Definition: $H_\psi := \phi_{\psi(\|p\|)}(N_D^* \setminus D)$ $\boxed{\text{flow handle}}$

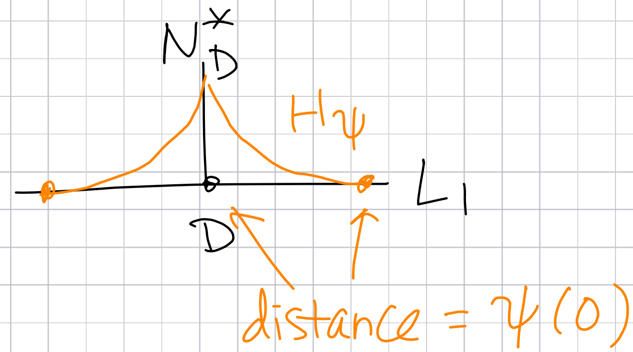
Geometric Description of ϕ_t:

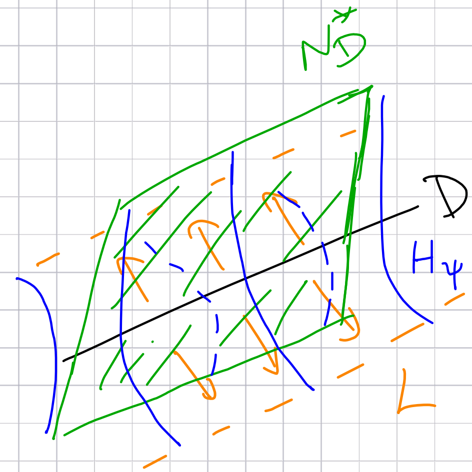
1) point (x, p) travels along direction pointed by p

2) $\phi_t(p)$ stays the (dual of) tangent vector of geodesic.

3) $\|\phi_t(p)\| = \|p\|$

Picture of H_ψ:





<u>Lemma</u>: If $\text{graph}(\psi) = (b(t), a(t))$,

$$\gamma(t) = (a(t), b(t)),$$

then $H_\psi = H_\gamma$ (Lalonde-Sikorav, Polterovich's handle)

Flow handles may go beyond inj. radius!

Motivation for flow handles:

1. Easy to define on clean intersection
2. Easy to construct Lag. Cobordism
3. Easy to compare with Dehn twists

Def: (Model Dehn Twist on $S \cong S^n$)

In T^*S, consider admissible $\psi(0) = 2\pi$ ($= \pi$ when $S = S^n$) length of simple geod.

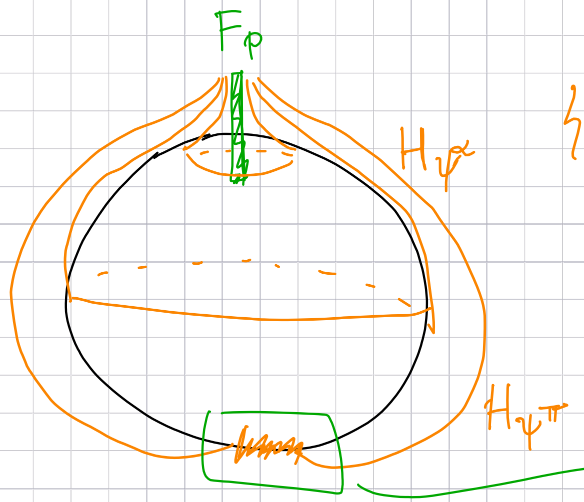
$\phi_{\psi(\|p\|)}$ smoothly extend to zero section

then τ_S is the extension.

$\Rightarrow$ flow handle is exactly the image of same flow

Example: $L = S \subset T^*S$, $S = S^n, \mathbb{RP}^n, \mathbb{CP}^n$

exotic sphere ...



$\{\psi^\alpha\}_{\alpha \in [0,\pi]}$, $\psi^\alpha(0) = \alpha$

D_p = opposite divisor

When $S = S^n$, $H_{\psi^\pi} = \tau_S(F_p) = S^n \#_p F_p$

$\Rightarrow$ Dehn. twist is isotopic to a surgery.

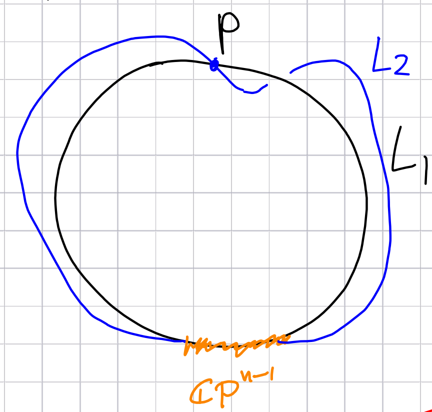
When $S = \mathbb{CP}^n$, $H_{\psi^\pi} = \mathbb{CP}^n \#_p F_p$

immersed Lag $S_{d} \to$

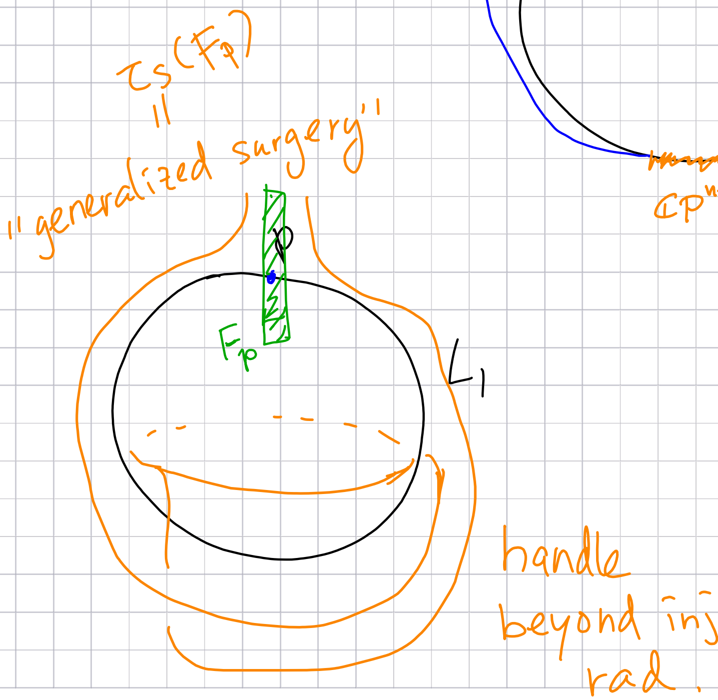
$$H_{\psi^{2\pi}} = \mathbb{CP}^n \#_{D_p} \mathbb{CP}^n \#_p F_p = \tau_{\mathbb{CP}^n} F_p$$

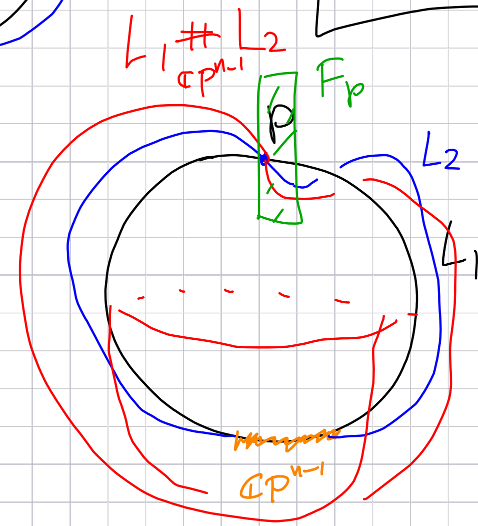
Example: $L_1 = \mathbb{C}P^n \ni \{p\}$, $L_2 = \mathrm{graph}(d(p,-))$ ← local pert. near p.

$M = T^*\mathbb{C}P^n$.



Pozniak: Clean-int. nbh $\simeq N_D^*$





Further Extensions (E_2-flow surgeries)

$D^m \subset L^n \overset{\text{Lag}}{\subset} M^{2n}$, $T^*L = E_1^{n-m} \oplus E_2^m$ (example: L = product mfd)

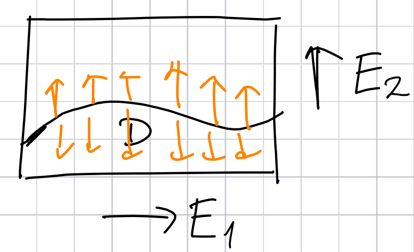
(implicitly identified with TL)

Assume $E_2|_D$ transverse to D.

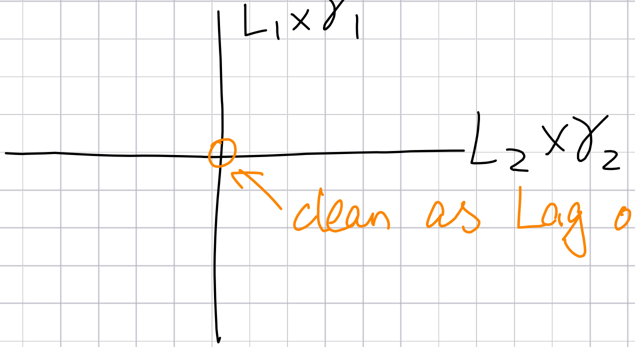
$\Rightarrow$ flow handle along E_2.

Concretely, assume $E_1 \perp E_2$, take Ham function

$$\tilde{\Psi}(\underset{\underset{L^n}{\in}}{x}, \underset{\underset{\in T_x^*L}{}}{p}) = \Psi(\|\pi_{E_2}(p)\|)$$



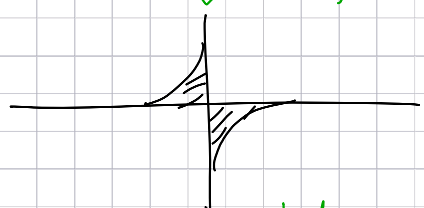
Constructing Lagrangian Cobordisms through flow surgeries



$L_1 \pitchfork L_2 = D$

clean as Lag on $M \times \mathbb{C}$

+ flow handle, $\mathbb{R}$-direction use product metric.



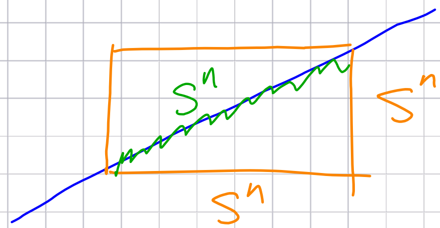
take upper half, straighten up

Biran-Cornea's trick, very adaptable.

First Main Example + Punchline

Consider $M \times M^-$, $L_1 = S^n \times (S^n)^-$, $L_2 = \Delta$

$(M, -\omega)$



$S^n \uparrow E_2$ (M^- - direction)

$\longrightarrow E_1$ (M - direction)

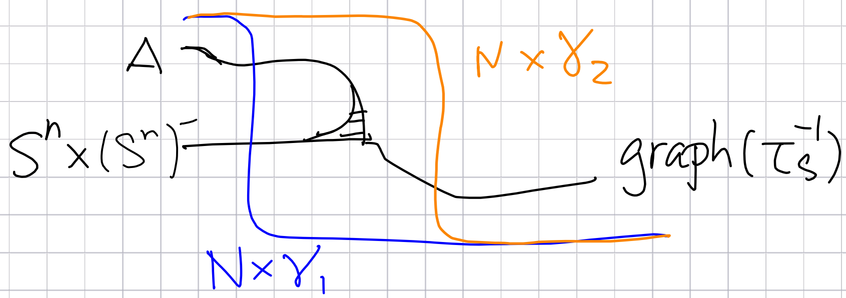
negative direction!

$T^*S^n \ni$ point $((x,p),(x,p)) \in \Delta$ flows according to $\Psi(\|p\|) = \Psi(\|\pi_1(p,p)\|)$ along E_2 direction

$\Rightarrow$ Image of flow handle is $((x,p), \tau_{S^n}^{-1}(x,p))$

$\Rightarrow (S^n \times S^n) \#_{\Delta_{S^n}} \Delta = \mathrm{graph}(\tau_{S^n}^{-1})$

Conclusion: $\exists$ Cobordism



(1) $N = L_0 \times L_1$, $\Rightarrow$ Seidel's exact sequence for Lag.

(2) $N = \text{graph}(\phi^{-1})$: $HF(\Delta, \text{graph}(\phi)) = HF(\phi)$

$\Rightarrow$ fixed point version (Easier to see if apply $(id \times \phi \times id)$ to cobordism & set $N = \Delta$)

Combine with MWW-functor.

Can be removed when LHS are product / graph of symp

$$\Phi : Fuk(M \times M^-) \to Fun(Fuk^{\#}(M), Fuk^{\#}(M))$$

$\Rightarrow$ functor-level cone $\Rightarrow$ object cone

$$\left(\begin{array}{c} S^n \boxtimes S^n \to Id \to \Phi_{\tau_{S^n}} \\ \downarrow \text{Feed } L \\ hom(S^n, L) \otimes S^n \to L \to \tau_{S^n} L \end{array} \right)$$

Corollary: $Symp_c(W)$ is split generated by Dehn twists along vanishing cycles, when $W \in ADE$.

(Mak-W., Keating)

$\star$ When $W = A_n$-Milnor fiber in $\dim_{\mathbb{R}} W = 4$,

$Symp_c(W)$ is generated (as a group / Ham) by U.C.

A few remarks:

1) New info: In monotone cases, Seidel's LES holds for coefficients $\mathbb{Z}/\mathbb{Z}_2$. Before it is known only for Novikov coefficients.

2) Seidel's exact sequence for general symp. mfds:

Ongoing $\rightarrow$ i) Lag. Cobordism for general symp. mfds

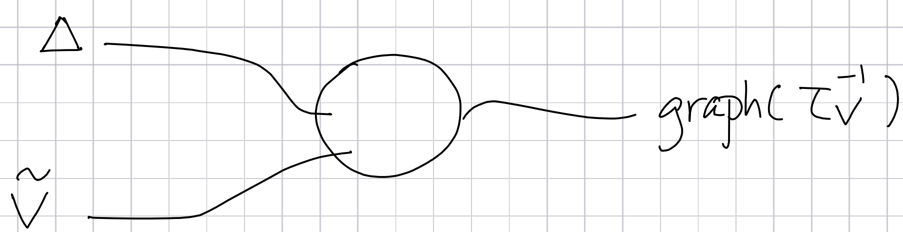
FOOO $\Rightarrow$ ii) Isom. $HF(\Delta, L_1 \times L_2) = HF(L_1, L_2)$

3) Also for spherically fibered coisotropic:

$$\begin{array}{ccccc} S^d & \longrightarrow & V^{2m+d} & \xhookrightarrow{\ i\ } & M^{2m+2d} \\ & & \downarrow{\scriptstyle \pi} & & \\ & & B^{2m} & & \end{array} \qquad i^*\omega_M = \pi^*\omega_B$$

(Think of $\underset{\substack{\cap \\ N}}{S^n} \times B \subset \overbrace{N \times B}^{M \,=}$)

$\Longrightarrow$



$\tilde{V} = \{(x, y) \in M \times M^- : \pi(x) = \pi(y)\}$

$\Longrightarrow$ Wehrheim - Woodward's family version.

Again, with stronger transversality results on quilts & cobordism theories, this yields family version on general symp. mfds.

4) (General Remarks on Lagrangian Surgeries)

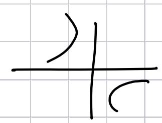
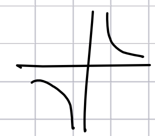
Algebraic "surgery" $\approx$ Cone of chain cx / objects:

$$\mathrm{Cone}(A \xrightarrow{c} B) = A[1] \oplus B$$

deg $= 0$!

c

i) This means the only categorically meaningful Lag surgeries are @ deg $= 0$ intersections $(CF^{0}(L_1, L_2))$

ii) Two Lag. surgeries  & 

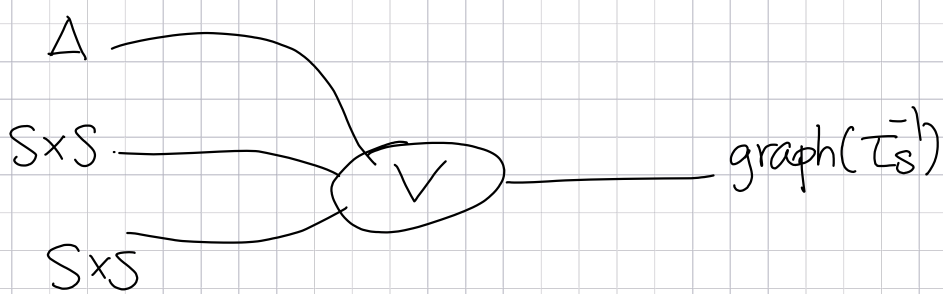
correspond to $CF(L_1, L_2)$ and $CF(L_2, L_1)$, resp.

$\Rightarrow$ to get meaningful objects needs grading shifts

$\Rightarrow$ resolving two intersections of $\neq$ deg creates problems (obstructions on $L_1 \# L_2$)

Theorem: (Mak–W.)

Given $S=\mathbb{CP}^n$, $\exists$ Lag. Cobordism for $M\times M^-\times\mathbb{C}$

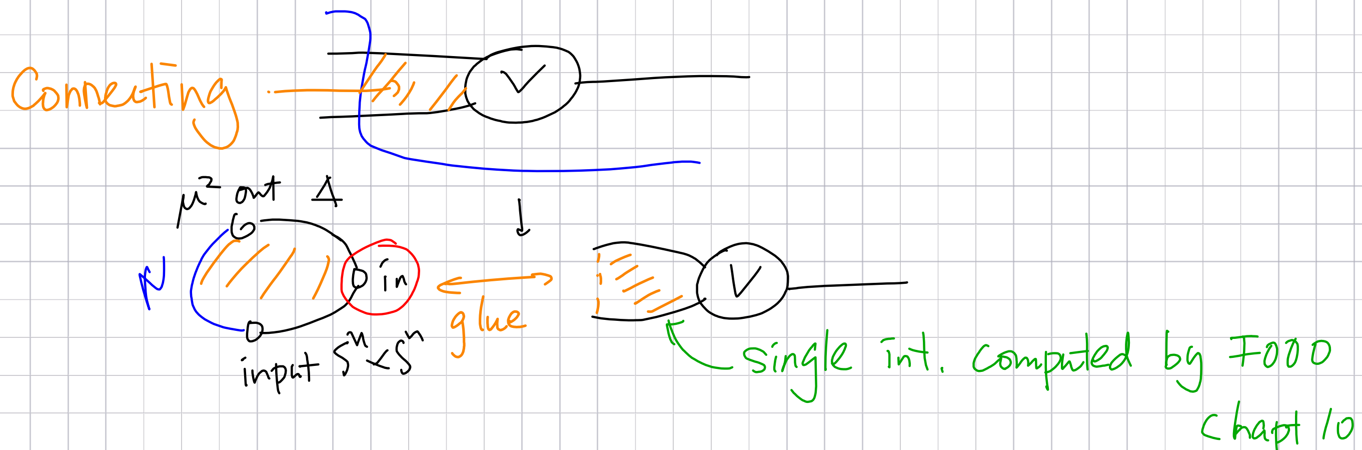


$$\Rightarrow \begin{array}{ccccccc} \mathrm{Cone}\,(S\times S[-2] & \to & S\times S & \to & \Delta) & = & \mathrm{graph}(\tau_S^{-1}) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \mathcal{E}^\vee\boxtimes\mathcal{E}[-2] & & \mathcal{E}^\vee\boxtimes\mathcal{E} & & \mathrm{Id} & & P_{\mathcal{E}} \end{array}$$

$\Rightarrow$ matching except for connecting maps

Similar situation for $S=\mathbb{RP}^n$, $\mathbb{HP}^n$, and family version.

Computation of Connecting Maps & Immersed Cobordisms.



Fact: $\mathrm{Cone}\left(A \xrightarrow{[C]} B\right) \underset{q\text{-iso}}{\simeq} \mathrm{Cone}\left(A \xrightarrow{[t \cdot C]} B\right)$

when t is invertible.

$\mathrm{rk}(HF^0(\Delta, S^n \times (S^n)^-)) = 1$! $\Rightarrow$ only need to verify connecting map $\neq 0$.

Rank-1 trick recovers Fooo's surgery exact seq.

Thm: (Fooo) $L_1 \pitchfork L_2 = \{p\}$ + technical conditions, then

$$\begin{tikzcd} L_1 \arrow[r, "{[p]}"] & L_2 \arrow[r] & L_1 \#_p L_2 \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$ is an exact triangle.

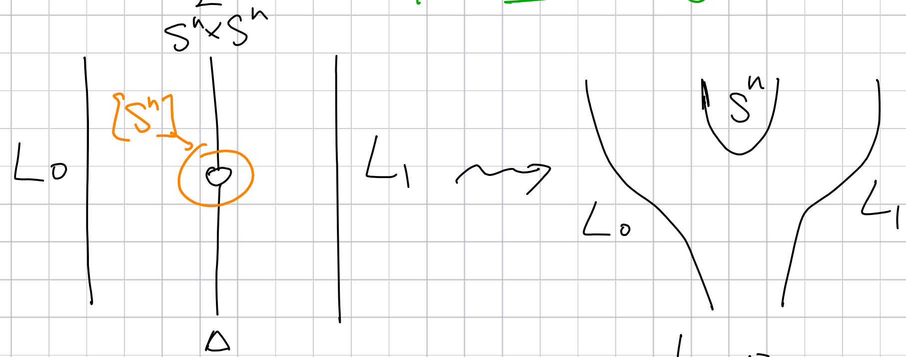
(Also due to Biran - Cornea)

* Can be extended to $L_1 \pitchfork L_2 = D$ a submfd.

$$\begin{tikzcd} L_1 \arrow[r, "{[D]}"] & L_2 \arrow[r] & L_1 \#_D L_2 \arrow[ll, bend left, "{[1]}"] \end{tikzcd}$$ with extra top. constraint.

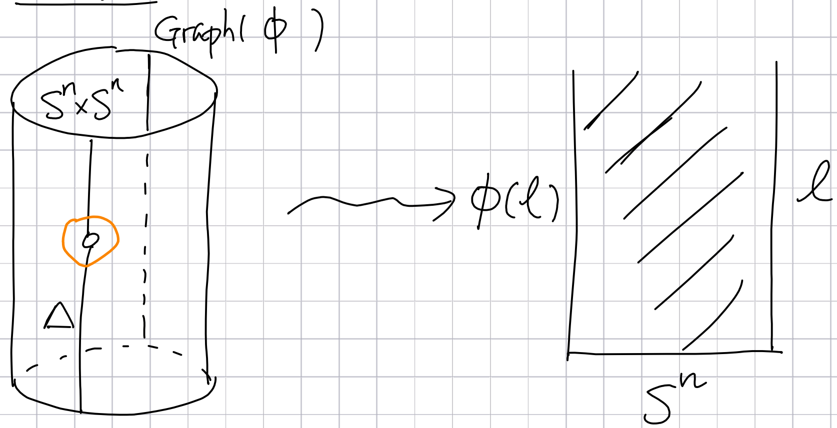
For Seidel's exact sequence: quilt unfolding

Lag. Case:



$=$ evaluation

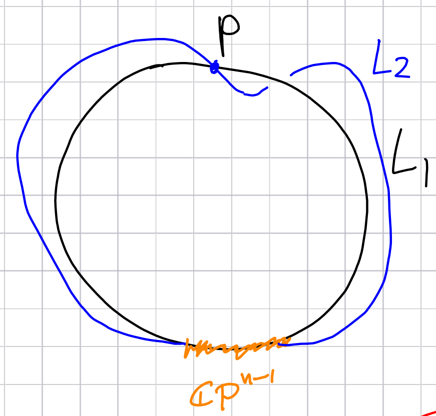
fixed point case:



$=$ open-closed for symp. ϕ.

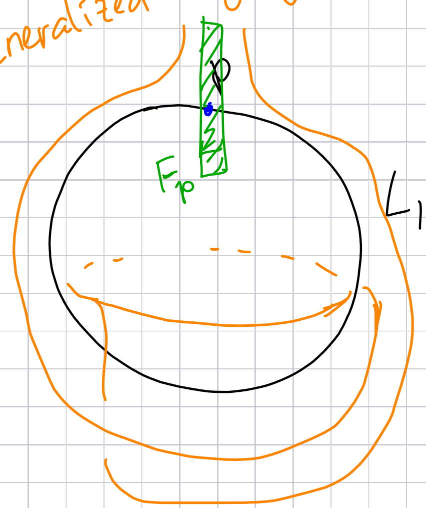
Recall: $L_1 = \mathbb{C}P^n \ni \{p\}$, $L_2 = \overbrace{\text{graph}(d(p,-))}$ ← local pert. near p.

$M = T^*\mathbb{C}P^n$



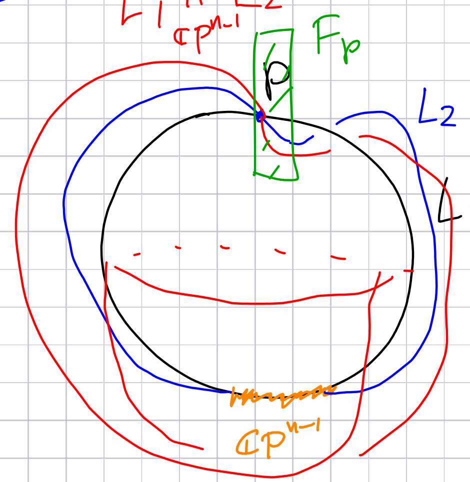
Pozniak: Clean-int. nbh $\simeq N_D^*$

$\tau_S(F_p)$
$=$
"generalized surgery"



handle beyond inj rad.

$L_1 \#_{\mathbb{C}P^{n-1}} L_2$



Claim: $F_p \#_p (\underbrace{L_1 \#_{\mathbb{C}P^{n-1}} L_2}_{S_{\perp\to}}) = \underbrace{\text{generalized surgery of } F_p}_{= \, \tau_{\mathbb{C}P^n} F_p}$

Theorem: (Mak–W.) when $L \pitchfork S = \{pt\}$

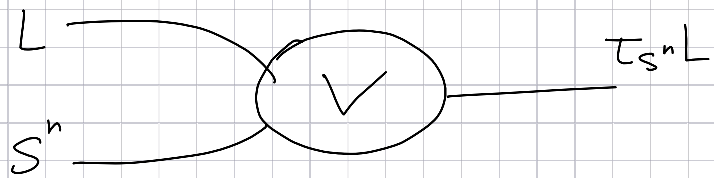
$$\begin{tikzcd} S[-2] \arrow[r, "{[\mathbb{C}P^{n-1}]}"] & S \arrow[d] \\ & S_{\perp\to} \arrow[ul, "{[1]}"] \end{tikzcd} \quad \& \quad \begin{tikzcd} S_{\perp\to} \arrow[r, "{[pt]}"] & L \arrow[d] \\ & \tau_S L \arrow[ul, "{[1]}"] \end{tikzcd}$$

* Works for all $S = \mathbb{R}P^n, \mathbb{C}P^n \ldots$ (different grading shift)

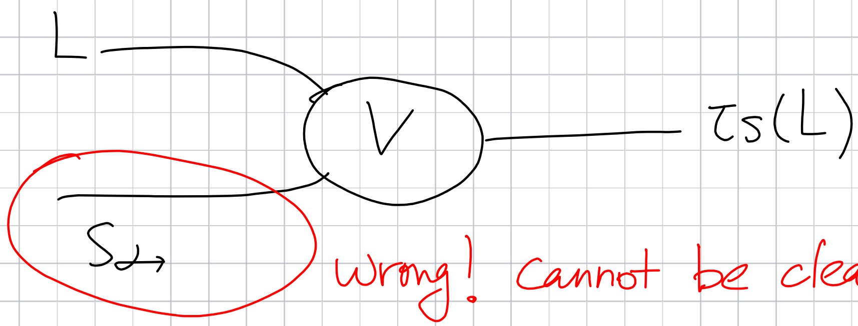
This matches Huybrechts–Thomas also for connecting maps.

Why do we need something new for immersed cobordism?

$S = S^n$, construction is straightforward:



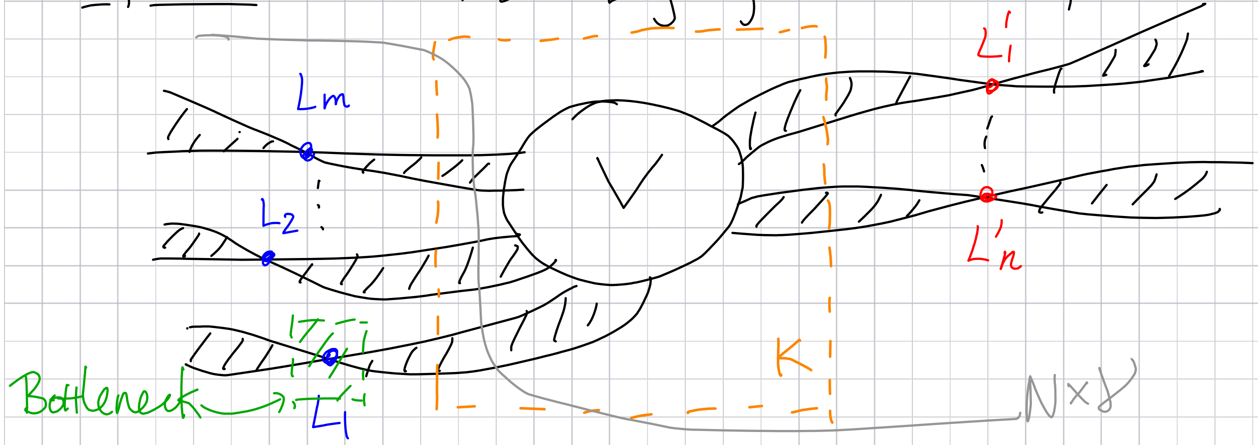
$S \neq S^n$, construction is harder:



wrong! cannot be clean self-int. !

exact!

Definition: (Immersed Lagrangian Cobordisms)



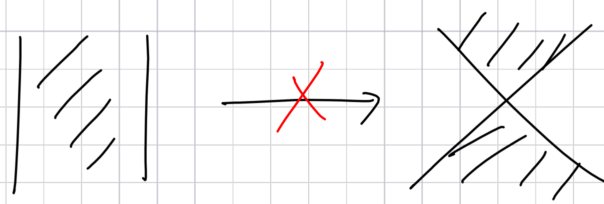
☆ Outside compact set K, each end of V should look like a bottleneck.

↳ An extension of Biran–Cornea's trick in embedded case

Lemma: $HF(N \times V', V)$ is well-defined and invariant under choices if

1) isotopy lies inside bottleneck.
2) near bottleneck, take product cx structures.

Key point:



$\Rightarrow$ compactness

exclude bubbles

Theorem: (Mak-W.) when L_i satisfy certain deg. restriction

$$\mathrm{Cone}(L_1 \to L_2 \cdots \to L_m) \simeq \mathrm{Cone}(L_1' \to \cdots \to L_n')$$

Some Prospects:

1) $\mathbb{RP}^n$ objects & twists. (Suggested by R. Thomas)

All stories told above holds for char $= 2$,

most hold for char $= 0$, except $\mathbb{RP}^n$!

When char $= 0$, Lag. $\mathbb{RP}^n \Longrightarrow$ spherical object,

Mystery 1. How to tell $\mathbb{RP}^n$ from S^n-objects?

Mystery 2. Will get contradiction when naively "upgrade" $\mathbb{RP}^n$-twist formula to $\mathbb{C}$-coefficient.

Answer: Blindsided by spin structures!

Expected: $(\mathbb{RP}^n, \mathfrak{s}_0) \# (\mathbb{RP}^n, \mathfrak{s}_1) \# L = \tau_{\mathbb{RP}^n} L$.

Definition: An $\mathbb{RP}^n$-pair in the derived cat $D^b(X)$ is a pair of spherical objects (S_-, S_+) such that $\mathrm{Hom}(S_-, S_+) = 0$

(+ more axioms)

$\Rightarrow$ $\mathbb{RP}^n$-twist $\in \mathrm{Aut}(D^b(X))$

(Ongoing with C.-Y. Mak)

2) Where to find $\mathbb{P}^n$-objects: Hyperkähler mfds

ex 1: $\mathbb{P}^n \subset X^{2n}$, $\Rightarrow$ $\mathcal{O}_{\mathbb{P}^n}$ is $\mathbb{P}^n$-object.

ex 2: $\pi : X \to \mathbb{P}^n$ Lag. fib. of irreducible hol. symp.
$+ H^*(X, \mathcal{O}_X) \cong H^*(\mathbb{P}^n, \mathbb{C})$.

$\mathcal{E}_Q$ $\quad \bigoplus \mathrm{Ext}^p(\mathcal{E}, \mathcal{E} \otimes \Omega^q) = \bigoplus H^p(\mathbb{P}^n, \Omega^q)$

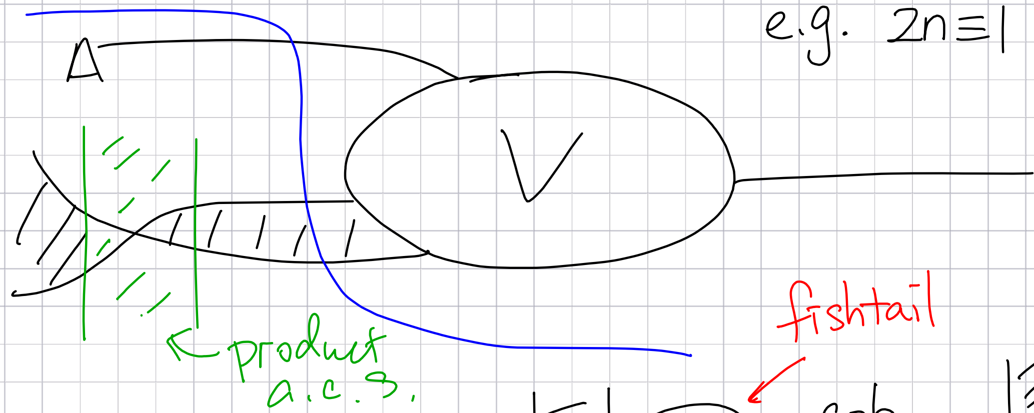
Then $\pi^*\mathcal{E}$ is $\mathbb{P}^n$-object.

Expectation: Interesting Lag. $\mathbb{CP}^n$'s should be found in mirrors of hyperkählers. (SYZ is usually nicely behaved)

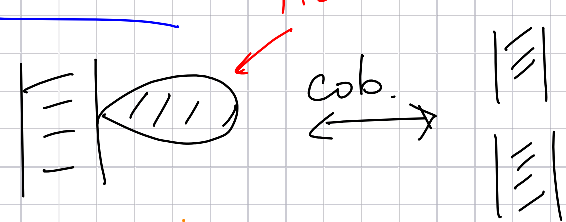
Lag. $\mathbb{CP}^n$-section?

3) Monotone Cases (in an early draft) (no grading obstructions to fishtails, e.g. $2n \equiv 1 \bmod \underline{N}$)

min. maslov



Potential Problem:



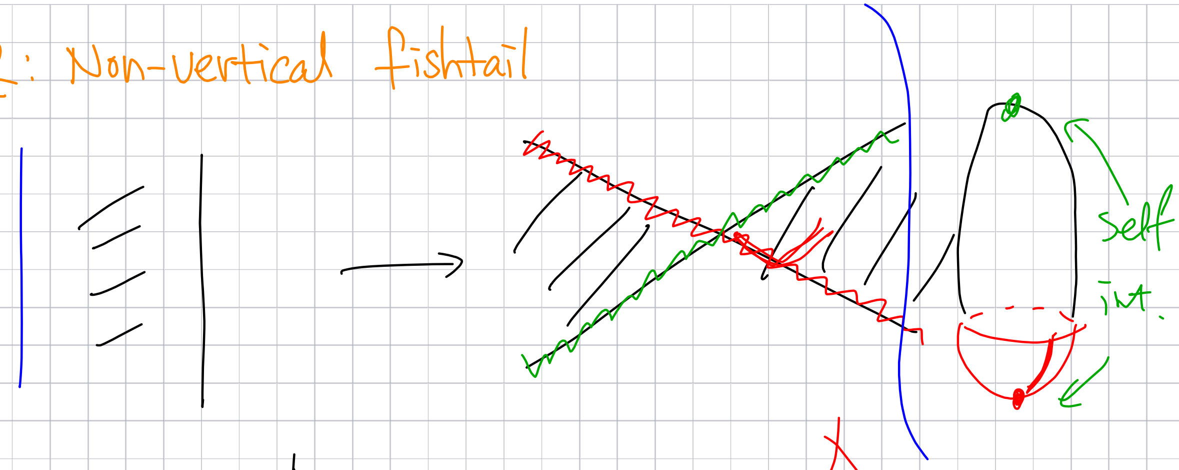
Case 1: Vertical fishtails

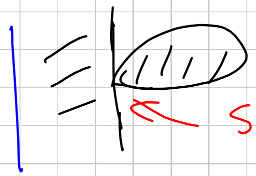
(1) exits $T^*\mathbb{C}P^n \Rightarrow$ Neck-Stretching

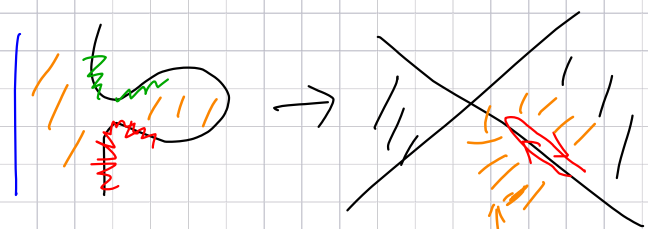
$\Rightarrow$ high virtual dim

(2) Inside $T^*\mathbb{C}P^n \Rightarrow$ absolute index

Case 2: Non-vertical fishtail



Assume  self-int ⇒ switch branch.

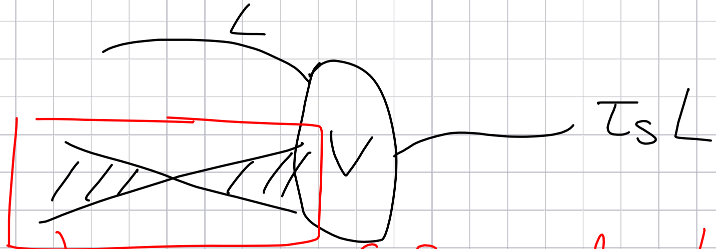
Before bubbling 

⇒ No such fishtails !

Contradiction!

More difficulties when continuation maps & chain homotopies are involved. ⇏ OK for 1 intersection.

In general,



$rk(CF(L,S))$ – copies of immersed spheres

Extremely difficult Problems:

gap #1

Well-def HF (no fishtail) but NOT

Well-def continuation map (—) but NOT

gap #2

Well-def chain homotopy (—)

Without appropriate grading gaps ⇒ needs explicit counts

Thank You !!