Viterbo conjecture

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Viterbo's conjecture

Conjecture (Viterbo)

If $X \subset (\mathbb{R}^{2n}, \omega_0)$ *is a convex set, then for any normalised symplectic capacity* c*,*

$$c(X) \leq (n!\,\mathrm{Vol}(X))^{\frac{1}{n}}$$

Moreover, we have equality iff X *is symplectomorphic to the ball.*

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Definition

A symplectic capacity is a function c which assigns to each symplectic manifold (X, ω) a number $c(X, \omega) \in [0, \infty]$, satisfying the following axioms :

- (Monotonicity) If $(X, \omega) \hookrightarrow (X', \omega')$, then $c(X, \omega) \leq c(X', \omega')$.
- (Conformality) If $r \in \mathbb{R}_{>0}$ then $c(X, r\omega) = rc(X, \omega)$.

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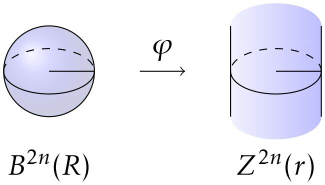
We say that a capacity c is normalized if

- $c\Big(B^{2n}(1)\Big) = c\Big(Z^{2n}(1)\Big) = 1.$

Symplectic capacities



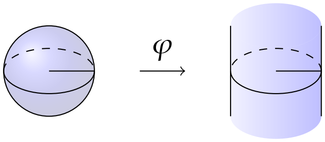
Theorem (Gromov)

$B^{2n}(R) \hookrightarrow^s Z^{2n}(r) \simeq D^2(r) \times \mathbb{R}^{2n-2}$ *iff* $R \leq r$.

Example (Gromov width)

$$c_{\mathrm{Gr}}(X,\omega) = \sup\{r \,|\, \exists (B^{2n}(r),\omega_0) \hookrightarrow^s (X,\omega)\}$$

Symplectic capacities



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Example (Cylindrical capacity)

$$c_Z(X,\omega) = \inf\{r \,|\, \exists (X,\omega) \hookrightarrow^s (D^2(r)\times\mathbb{R}^{2n-2},\omega_0)\}$$

Symplectic capacities

Example

- The first *Ekeland-Hofer capacity*, c_1^{EH},
- The *Hofer-Zehnder capacity*, c_{HZ},
- The *Viterbo capacity*, c_{SH},
- The first *capacity from S^1-equivariant symplectic homology*, c_1^{CH},
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Lemma

If $X \subset \mathbb{R}^{2n}$ is a convex set, then

$$\left(c_{\mathrm{Gr}}(X)\right)^n \leq n!\,\mathrm{Vol}(X)$$

Weak Viterbo conjecture

Theorem

If $X \subset \mathbb{R}^{2n}$ is a convex set, then

$$c_1^{EH}(X) = c_{HZ}(X) = c_1^{CH}(X) = c_{SH}(X) = T_{\min}$$

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Conjecture (Weak Viterbo conjecture)

If $X \subset \mathbb{R}^{2n}$ is a convex set, then

$$\left(T_{\min}\right)^n \leq n!\,\mathrm{Vol}(X)$$

Conjecture (Strong Viterbo conjecture)

All normalized symplectic capacities coincide on convex sets in $\mathbb{R}^{2n}$.

$$\text{Strong V. Conj.} \quad \Rightarrow \quad \text{V. Conj.} \quad \Rightarrow \quad \text{Weak V. Conj.}$$

Strong Viterbo conjecture

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$$\text{Strong V. Conj.} \quad \Rightarrow \quad \text{V. Conj.} \quad \Rightarrow \quad \text{Weak V. Conj.}$$

Idea

Either, prove that for any convex set $X \subset \mathbb{R}^{2n}$

$$c_{\mathrm{Gr}}(X) = c_Z(X)$$

Or, find a convex set $X \subset \mathbb{R}^{2n}$ *such that*

$$\left(T_{\min}\right)^n > n!\,\mathrm{Vol}(X).$$

Corollary 1

Theorem (ABHS)

There exists a C^3-neighborhood $\mathcal{A}$ of the space of Zoll contact forms on S^3 such that

$$T_{\min}(\alpha)^2 \leq \mathrm{Vol}(S^3, \alpha \wedge d\alpha) \forall \alpha \in \mathcal{A}$$

with equality holding if and only if α is Zoll.

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Corollary

Let c be either the Ekeland-Hofer or the Hofer-Zehnder capacity on $(\mathbb{R}^4, \omega_0)$. There exists a C^3-neighborhood $\mathcal{C}$ of the set $\mathcal{B}$ of all convex domains in $\mathbb{R}^4$ whose closure is symplectomorphic to a closed 4-ball, within the set of all convex smooth domains in $\mathbb{R}^4$ such that

$$c(C)^2 \leq 2\,\mathrm{Vol}(C) \quad \forall C \in \mathcal{C},$$

with equality holding if and only if C belongs to $\mathcal{B}$.

Quick recap

$\Sigma \subset \mathbb{R}^{2n}$ star-shaped bounded domain with smooth boundary. Then

$$\lambda_0 = \frac{1}{2} := \sum_{i=1}^{n} x^i dy^i - y^i dx^i$$

restricts to a contact form on $\partial\Sigma$.
$h_\Sigma : S^{2n-1} \to \mathbb{R}$ smooth positive, $\Sigma = \{rz | z \in S^{2n-1}, 0 \leq r \leq h_\Sigma(z)\}$
$\varphi : S^{2n-1} \to \partial\Sigma, \quad z \mapsto h_\Sigma(z)z$

$$\varphi^\star(\lambda_0|_{\partial\Sigma}) = h_\Sigma^2 \lambda_0|_{S^{2n-1}}$$

α for $\xi_0 \Leftrightarrow$ star-shaped domains

Quick recap

$$\begin{aligned}
\mathrm{Vol}(S^3, \alpha \wedge d\alpha) &= \mathrm{Vol}(S^3, \varphi^\star(\lambda_0|_{\partial\Sigma}) \wedge d\varphi^\star(\lambda_0|_{\partial\Sigma}) \\
&= \mathrm{Vol}(\partial\Sigma, \lambda_0|_{\partial\Sigma}) \wedge d\lambda_0|_{\partial\Sigma}) \\
&= \mathrm{Vol}(\partial\Sigma, (\lambda_0 \wedge d\lambda_0)|_{\partial\Sigma}) \\
&= \mathrm{Vol}(\Sigma, \omega_0 \wedge \omega_0) \\
&= 2\,\mathrm{Vol}(\Sigma, dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2) \\
&:= 2\,\mathrm{Vol}(\Sigma)
\end{aligned}$$

Convexity vs dyn convex

Convexity is not a symplectic notion. HWZ introduced the dynamical convexity

Definition

A contact form α on S^{2n-1} is *dynamically convex* if it is non-degenerate and all closed Reeb orbits have Conley-Zenhder index at least $n+1$.

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Theorem (HWZ)

Every convex domain is dynamically convex.

Conjecture

Every dynamically convex domain is symplectomorphic to a convex domain.

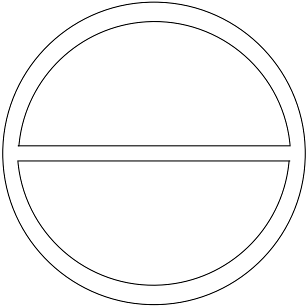
Theorem

For every $\epsilon > 0$ there is a dynamically convex contact form α on S^3 such that

$$2 - \epsilon < \frac{T_{\min}^2}{\mathrm{Vol}(S^3, \alpha \wedge d\alpha)} < 2$$

In particular, the supremum of the systolic ratio over all dynamically convex contact forms on S^3 is at least 2.

“Proof”



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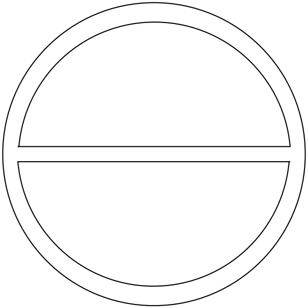

Dichotomy

Either Viterbo conjecture or dynamically convex $\simeq$ convex

toric domains

Definition

A toric domain $X_\Omega \subset \mathbb{C}^n$ is a set of the form $X_\Omega = \mu^{-1}(\Omega)$ where Ω is a domain in $[0,\infty)^n$ and

$$\mu : \mathbb{C}^n \to [0,\infty)^n \quad \mu(z_1,\ldots,z_n) = (\pi|z_1|^2,\ldots,\pi|z_n|^2)$$

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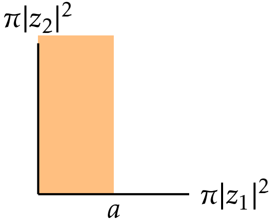
Example (Cylinder)



$Z(a) := \left\{(z_1,z_2) \in \mathbb{C}^2 \,|\, \pi|z_1|^2 \le a\right\}$

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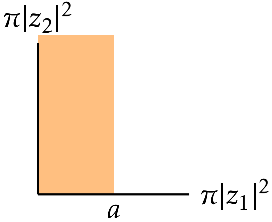

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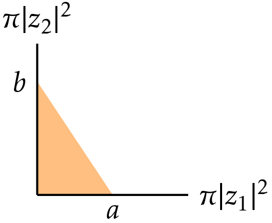
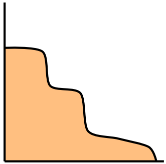
Example (Ellipsoid)



$E(a,b) := \left\{(z_1,z_2) \in \mathbb{C}^2 \,\middle|\, \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\right\}$

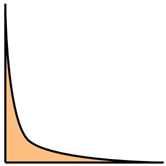
Definition

If Ω' is a domain in $[0,\infty)^{n-1}$ and $f:\Omega'\to[0,\infty)$ is a non-increasing function A toric domain X_Ω is called *dynamically convex* if there exists a domain $\Omega'\subset[0,\infty)^{n-1}$ and a non-increasing, continuous, piecewise smooth function $f:\Omega'\to[0,\infty)$ such that Ω is the relatively open set in $[0,\infty)^n$ bounded by the coordinate hyperplanes and the graph of a non-increasing function.



Remark

Dynamically convex toric domains are not necessarily convex



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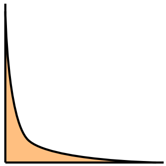

Proposition

A toric domain X_Ω is dynamically convex if, and only if, there is a sequence of domains Ω_ϵ converging to X_Ω in the C^0-topology and dynamically convex contact forms λ_ϵ on $\partial\Omega_\epsilon$ converging to the stardard Liouville form $\lambda_0|_{\partial X_\Omega}$.

strong Viterbo conjecture

Theorem

For a 4-dimensional dynamically convex toric domain X_Ω all symplectic capacities coincide.

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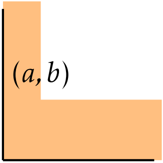
Define the non-disjoint union of two cylinders as

$$Z_2(a,b) := \{(z_1, z_2) \in \mathbb{C}^2 \,|\, \pi|z_1|^2 \leq a \text{ or } \pi|z_2|^2 \leq b\}$$



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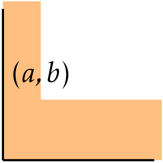
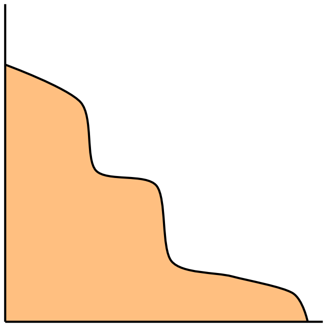


Lemma

$$Z_2(a,b) \overset{s}{\hookrightarrow} Z(a+b)$$

proof



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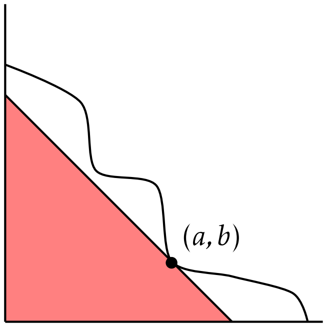

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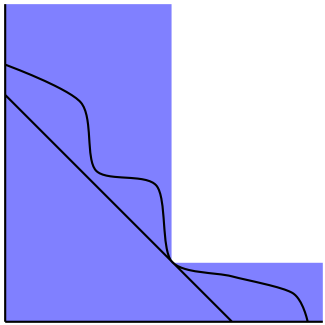

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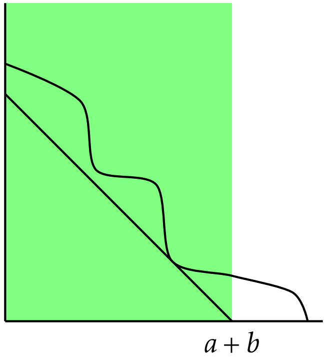
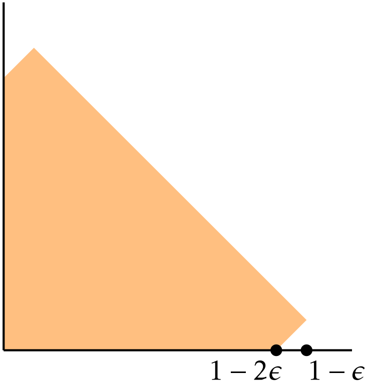


convexity ?



$c_{\mathrm{Gr}} = 1-2\epsilon$ (?), $\qquad c_Z = 1-\epsilon$

But (weak) Viterbo conjecture is still true !