Symplectic / Lag. Configurations: Existence and stability

Motivation: minimal genus problem

given $A \in H_2(M^4; \mathbb{Z})$, what is the minimal genus of an embedded representative of A?

Thom conjecture: If M is symplectic, and A is represented by symp surf. Σ, then $g(\Sigma) = g_{min}$.

(Oszváth - Szabó, Kronheimer - Mrowka)

Question: Given an explicit homology class, how to compute the minimal genus explicitly, or, estimate it?

Particular interests lies in to determine when A has a genus 0 symplectic representative: many surgery operations are available.

focus

Testground: $\boxed{\text{Rational}}$ / Ruled surfaces.

$M =$

$\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2 : \{H, E_1, \cdots, E_n\}$

Basic constraints:

$$c_1^{\omega}(A) = A^2 + 2 - 2g(A)$$

1) This equality is almost all one needs if $Gr(A) \neq 0$.

2) Classification for $(-1), (-2), (-3)$ -spheres are complete in rational / ruled (Li-W., Borman-Li-W.)

Main Theorems: (Dorfmeister-Li-W.)

1) $A \in H_2(M;\mathbb{Z})$ is represented by (-4)-symp. sphere iff it has a smooth sphere rep. and $\omega(A) > 0$. (classes explicitly described)

2) (existence of plumbings)

For classes $\xi_i = E_i - E_{i+1}$, there is a A_n-symp. (Lag.) conf. $\{S_i\}$, s.t. $[S_i] = \xi_i$ if $\omega(\xi_i) > 0$ $(\omega(\xi_i) = 0)$.

A_n:

$$\underset{\substack{\xi_1\\ \parallel \\ E_1-E_2}}{\bullet} - \underset{\substack{\xi_2\\ \parallel \\ E_2-E_3}}{\bullet} - \bullet \cdots \bullet - \underset{\substack{\xi_n\\ \parallel \\ E_n-E_{n+1}}}{\bullet}$$

Similar results hold for all ADE-type plumbings for (-2)-symplectic spheres.

$\ast$ Combining the two methods one should be able to find a large class of symplectic plumbing configs for surgery operations.

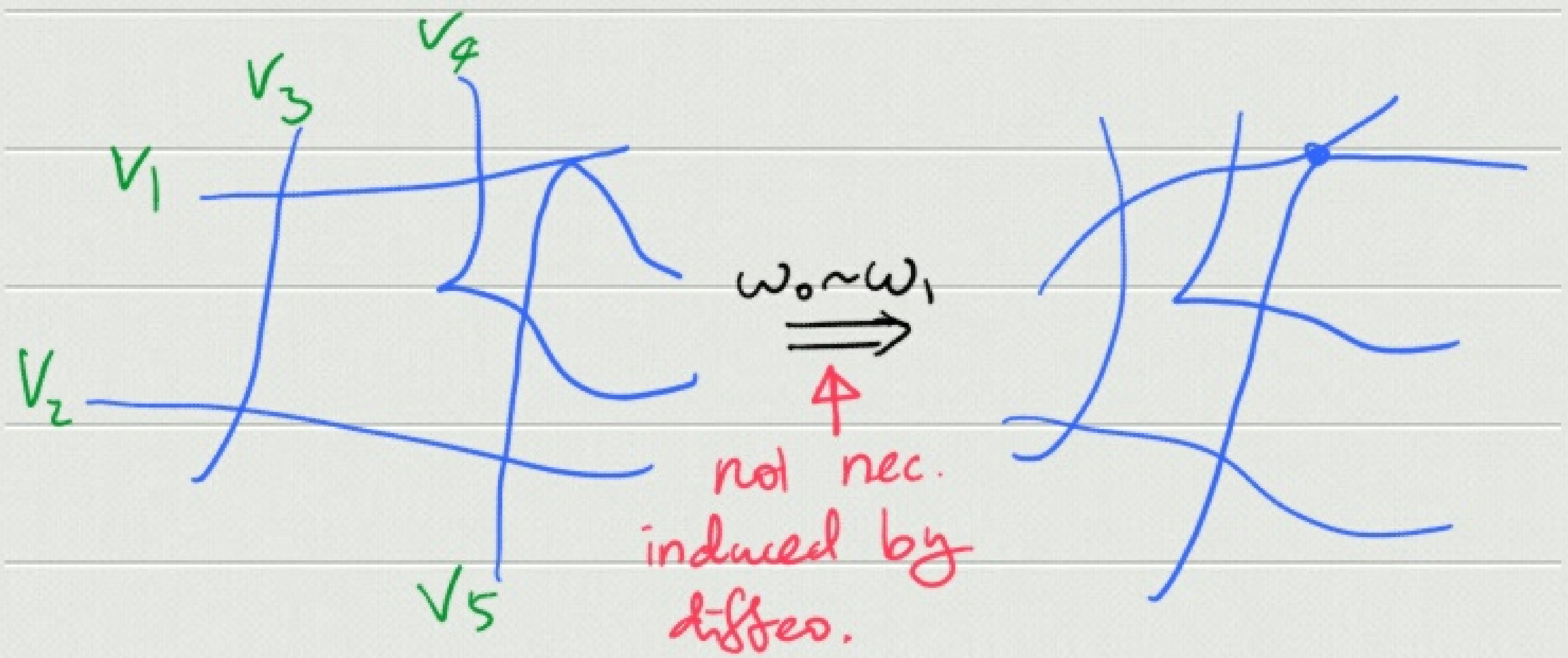
<u>Main Ingredients for the proof</u>:

1) Flexibility of symplectic forms.

The statement of main theorems want to find symplectic reps. for all symplectic forms where pairing are positive.

<u>Upshot</u>: It is sufficient to find symplectic rep. for one symplectic form in a deformation class.

Theorem (D-L-W.) Given any symplectic conf. Γ with pos. self-int. in (M,ω) $b^+(M)=1$, then there is a symplectic Γ' for any (M,ω') if $\omega \overset{\text{def}}{\sim} \omega'$.



Sketch of proof for main theorem:

Classification for (-4)-spheres has two parts:

1) Find all possible hom. classes that admit emb. rep.
2) Find actual symp. rep for some symplectic form.

We will focus on 2), but 1) gives classes (up to diffeomorphism)

i) $E_1 - E_2 - E_3 - E_4$

ii) $a \cdot (3H - E_1 - \cdots - E_9) - 2E_{10}$

For type (i) rep. easily constructed: iterate blow-up.

For type ii), consider elliptic fibration of $E(1)$.

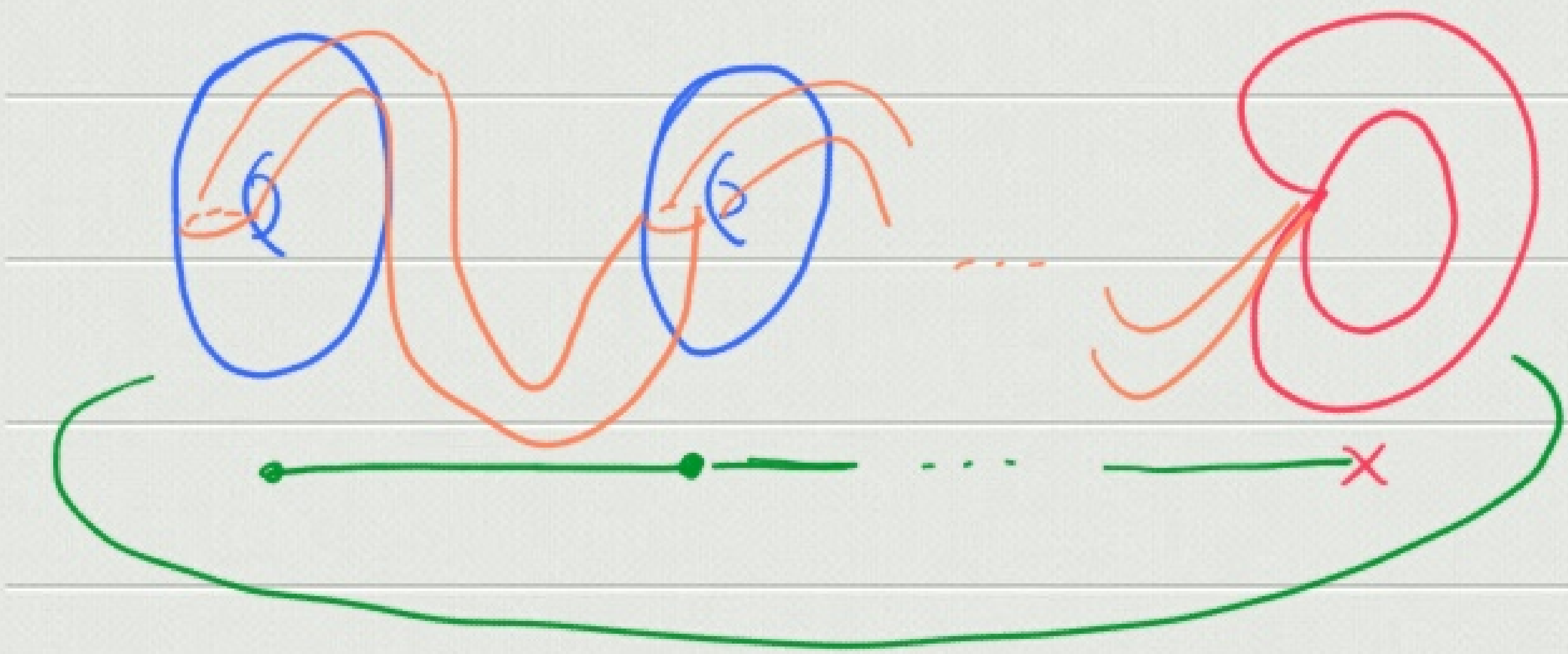
neighborhood of sing. fiber

Fiber class $= 3H - E_1 - \cdots - E_9$, class ii) obtained by circle summing $(a-1)$ generic fiber + 1 sing. fiber.

Symplectic parallel transport:

Given a Lef. fibration, $\exists$ natural conn. $\Rightarrow$ parallel transport.

In dim=4, when parallel transport of curve gives a Lag. chain, if a vector field $\perp$ to the curve add $\Rightarrow$ Symplectic chain



This is a tilted transport, doubling

$\Rightarrow$ symplectic immersed sphere $a(3H-E_1 \cdots - E_9)$.

$\Rightarrow$ blow-up sing. gives $a(3H-E_1 \cdots - E_9) - 2E_{10}$

2) Embedded A_n-sing.

For symplectic, deform all exceptional $E_1, \cdots, E_{n+1}$ to a small size

$\Rightarrow$ Construct in $B^4 \# (n+1)\overline{\mathbb{CP}}^2$.

(easy)

Lagrangian case : conifold transition.

reduce to symp. case.

* need to study relation between conifold transition and cptly supp. symp. deformation.

Result: One may undo conifold transition in a nbh of ADE-conf. at cost of changing symplectic form on ∂(nbh).